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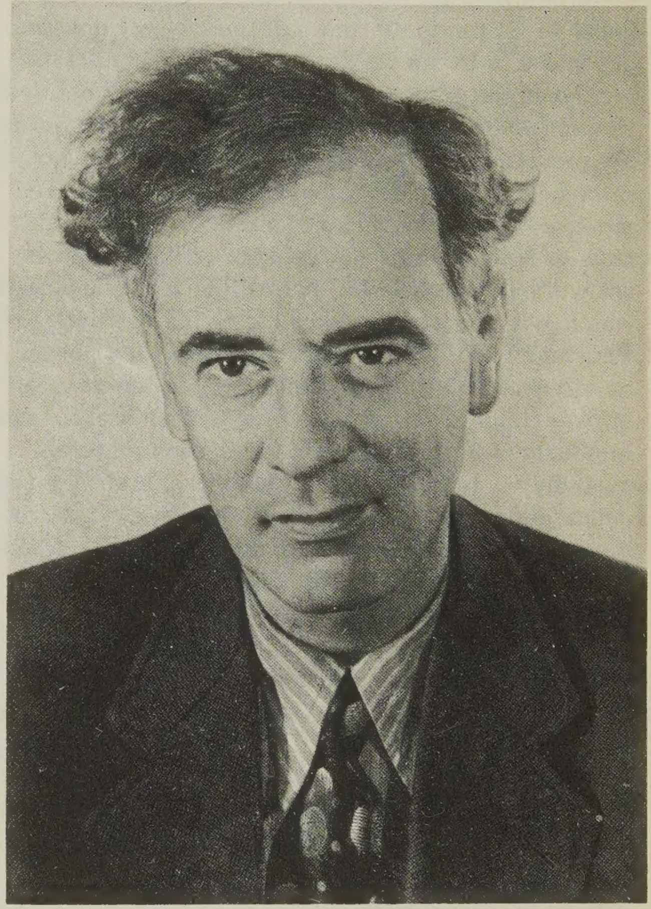
ACADEMICIAN LEV DAVIDOVICH LANDAU

(On the fiftieth anniversary of his birth)



LEV Davidovich Landau, one of our greatest theoretical physicists, completed his fiftieth year of life on January 22.

L. D. Landau was born in 1908 in the city of Baku. His mathematical talents began to appear at a very early age. He can scarcely remember not being able to differentiate and integrate. At the age of fourteen he was admitted as a student in the Physics Faculty of the University of Leningrad, where he completed his studies in 1927. This time marks the beginning of Lev Davidovich's intense scientific activity.

Lev Davidovich began his scientific work in the Leningrad Physico-technical Institute; two years later, in 1929, he received the opportunity to spend considerable time in study abroad. For a year and a half he worked at the Institute for Theoretical Physics in Copenhagen, and also in Germany, England, and Switzerland. Discussions and collaboration with the leading theoretical physicists of that time aided in the development of the wide scope of his interests as a physicist; this is particularly true of his association with Niels Bohr, and Lev Davidovich regards himself as a pupil of Bohr.

After his return from abroad he did not remain long in Leningrad, and moved to Khar'kov, to the Ukrainian Physico-technical Institute, where he was the head of the theoretical section for five years. This same place saw the beginning of his pedagogical activity, in the Physico-mechanical Faculty of the Khar'kov Institute of Machine Construction and at the University of Khar'kov. Lev Davidovich's talent as a teacher was apparent from his early years, and pupils began to gather around him. Thus his time in Kharkov already saw the beginning of the broad scientific school of Lev Davidovich Landau.

Lev Davidovich's main pedagogical principle is his conviction that independent creative work in any field of theoretical physics must begin with a sufficiently deep mastery of all its branches. For this purpose he developed a special program which has become widely known among young physicists under the name of the "theoretical minimum".

In 1937 Lev Davidovich moved to Moscow, where he is the head of the theoretical section of the then newly organized Institute for Physical Problems of the Academy of Sciences of the U.S.S.R. He is continuing his pedagogical activity at the University of Moscow; for a number of years he was a professor in the Physico-technical Faculty, and is at present a professor in the Physics Faculty.

The important scientific and scientific-social services of L. D. Landau have found their due recognition in the Soviet Union: in 1946 he was elected an active member of the Academy of Sciences of the U.S.S.R., and he has been awarded the Stalin Prize three times and has been honored with Soviet decorations. Outside his own country Lev Davidovich has been elected to membership in the Danish and Dutch academies of science.

The scientific activity of L. D. Landau amazes one by its extraordinary versatility. In our age of specialization, when there remain almost no physicists of universal interests, he not only feels at home in all the main fields of theoretical physics, but also has to his credit a number of important contributions in hydrodynamics and aerodynamics. His approach to the study of theoretical physics is characterized by the fact that he does not separate the theory from natural reality, nor from experiment, so that he remains close to experimental physics, on which he has a great influence. It is not without significance that at the weekly seminar which Lev Davidovich conducts at the Institute for Physical Problems reports are presented not only on theoretical researches but also on the results of experimental work on the most varied problems in physics. Participants in the seminar are repeatedly amazed to see Lev Davidovich show equal enthusiasm and thorough knowledge in discussing, for example, the energy spectrum of the electrons in silicon, directly after dealing with the properties of the so-called "strange" particles.

The scientific works of L. D. Landau are so numerous and of such varied natures that it is impossible to give in a few pages any sort of complete survey and characterization of all his contributions, which have been of great importance in all fields of theoretical physics, beginning with hydrodynamics and ending with quantum field theory. We shall confine ourselves to mentioning only the most significant of his publications.

In 1927, in connection with the problem of radiative damping in quantum mechanics, Lev Davidovich was the first to introduce the concept of the density matrix (in the energy representation).[1]

In 1930 he published his classic work on the diamagnetism of electrons in a metal (Landau diamagnetism),[2,3] which later led to the explanation of the variations of the magnetic susceptibilities of metals at low temperatures in strong magnetic fields (de Haas-van Alphen effect). Another important contribution to the development of present ideas about the magnetic properties of matter is contained in a paper published in 1935 (together with E. M. Lifshitz),[4] in which the thermodynamic theory of domains in ferromagnetic substances and the foundations of the theory of ferromagnetic resonance were given for the first time. Credit is also due to Lev Davidovich for first introducing into physics the idea of the antiferromagnetic ordering of magnetic moments, as a special phase of a substance, which exists at low temperatures and goes over into the paramagnetic condition at a point of phase transition.[5]

During this same time Lev Davidovich put forward the important idea of the autolocalization of electrons in a crystalline lattice,[6] which subsequently influenced S. I. Pekar's development of his polaron theory.

In addition to papers containing new theoretical ideas, Lev Davidovich's creative activity in science is also characterized by papers devoted to the solution of difficult theoretical problems by brilliant mathematical flank attacks. An example is his general method for solving quasi-classical collision problems, published in 1932.[7]

Lev Davidov made a fundamental contribution to thermodynamics by developing his theory of phase transitions of the second kind, which reveals their deep connection with the symmetry properties of a substance.[8] The methods he developed were first applied to transitions of a structural character, and later were applied with success to the study of piezoelectricity, ferroelectricity, antiferromagnetism, and so on.

The name of L. D. Landau is associated with the creation of a theory of the superfluidity of liquid helium,[9] which he developed with a completeness that makes possible an exhaustive explanation of the numerous phenomena in liquid helium, and which has now won its way to general acceptance. For this work and for his work on the theory of phase transitions Lev Davidovich was awarded the Stalin prize in 1941.

Landau's contribution to another problem of low-temperature physics — superconductivity — is primarily comprised in his theory of the laminar structure of the intermediate state.[10]

Quite recently Lev Davidovich has developed a new general theory of the quantum Fermi liquid, which has predicted a number of interesting phenomena in liquid He^3 (Ref. 11) which still await experimental confirmation.

Turning to the field of nuclear physics, we must note first of all a paper by Lev Davidovich in 1937[12] in which, starting from an idea of N. Bohr, he laid the first foundations of the quantitative statistical theory of the nucleus, and in particular obtained a connection between the neutron width and the separation of the levels. In 1938 Landau (together with

G. Rumer) gave an elegant mathematical theory of the production of electron-photon showers;[13] the procedure developed in this paper subsequently became the basic method in the study of cascade processes. In 1953, starting from an idea of Fermi, he constructed a consistent hydrodynamical theory of multiple production of particles in collisions of fast particles.[14]

In the last few years Lev Davidovich has published a series of papers on quantum electrodynamics (together with A. A. Abrikosov and M. I. Khalatnikov), in which the asymptotic forms of the propagation functions at high energies are determined.[15] Starting from these results,[16] I. Ia. Pomeranchuk subsequently arrived at a most important conclusion, that it is impossible to construct a relativistic theory of the strong interactions as point interactions.

Only a year ago, in connection with the exciting question of the nonconservation of parity, Lev Davidovich proposed the ingenious hypothesis of the conservation of the combined parity,[17] which at present is still awaiting experimental confirmation.

In concluding this brief survey, in order to show the breadth of L. D. Landau's scientific interests, we shall mention a few more papers that stand a bit aside from the others: the theory of the fine structure of the Rayleigh scattering (together with G. Placzek);[18] the now widely used kinetic equation for the case of Coulomb interaction between the particles;[19] the theory of the structure of shock waves at great distances from their place of origin.[20]

The breadth of Lev Davidovich's grasp of contemporary physics is even more convincingly shown by the course of theoretical physics which he has written together with E. M. Lifshitz.[21] Taken together, these books are a fundamental treatise on theoretical physics. In originality of exposition and broad grasp of the material they are unprecedented in the whole world-wide literature of physics, and so have attained wide popularity not only in our country but also abroad.

The contribution for which theoretical physics is indebted to Lev Davidovich is not exhausted by his own scientific writings. We have already spoken of another side of his activity — his founding of a broad school of Soviet theorists.

His inextinguishable enthusiasm for science, his acute criticism, his talent and clarity of thought attract many young people to Lev Davidovich. The number of those, both young and mature scientists, who turn to Dau (as his pupils and associates have come to call him) is very large. Lev Davidovich's criticism is hot and merciless, but behind this outer sharpness is hidden devotion to high scientific principles and a great human heart and human kindness. Equally sincere is his wish to aid the success of others with his criticism, and equally warm is his expression of approval.

Together with all the physicists of our country, the editors of the JETP place a high value on the great contribution which Lev Davidovich has made to the development of our science, and we send to him sincere wishes for many years of health, happiness, and creative success in the service of Soviet science.

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Translated by W. H. Furry
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REFLECTION OF SLOW ELECTRONS FROM THE SURFACE OF PURE TUNGSTEN AND FROM TUNGSTEN COVERED WITH THIN FILMS. II

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Kiev State University

Submitted to JETP editor May 8, 1957

J. Exptl. Theoret. Phys. (U.S.S.R.) 33, 7-13 (1958)

Studies have been made of the reflection of slow electrons from monocrystalline tungsten and from tungsten crystals on which a layer of tungsten has been deposited by evaporation, under conditions of ultra-high vacuum ($p < 10^{-9}$ mm Hg) to ensure a clean surface. Anomalous changes occur in the reflection coefficients when thin layers of Ba, BaO, and Ba-O are applied to a tungsten single crystal; a possible explanation is discussed. The experimental results are compared with calculations for the case of slow electrons reflected from a potential barrier possessing a maximum. The diffraction of slow electrons from monocrystalline tungsten has been studied, and also the effect of depositing thin films of barium.

A previously published paper[1] contained a report of the results of studies on the reflection of slow electrons from the surface of a polycrystalline tungsten ribbon, and from thin films of barium and oxygen upon the tungsten. The experiments were carried out in sealed-off tubes at total pressures of the order of 10^{-9} mm Hg, and with equivalent pressures of 10^{-10} to 10^{-11} mm Hg in the adsorption regions, thus preventing any distortion of the results by the adsorption of residual gas. The reflection coefficient R for slow electrons from the surface of the tungsten target showed an unusual type of dependence on the energy of the incident primary electrons, V_p; as the energy increased, so did the reflectivity. Meyer[2] has obtained similar results with tungsten. This behavior of the reflection coefficient cannot be explained as an interaction of the electrons with the potential barrier at the metal-vacuum boundary, for any reasonable barrier shape must give a decrease in reflectivity as the energy increases. In these experiments there was still an unknown factor in the polycrystalline nature of the sample, which could conceivably have altered the results, both for the pure tungsten and for the systems with thin films which were studied. For this reason, experiments have been carried out on monocrystalline tungsten targets. The results of these experiments, and of further studies of the effects of thin films on the reflection of slow electrons, are reported in this paper.

MEASUREMENT TECHNIQUES

Reflection coefficient measurements were carried out in sealed-off tubes whose construction was basically the same as that already described[1] (cf. Fig. 1 of Ref. 1). Unlike the ones used previously, these tubes used a glass sphere 35 mm in diameter, coated internally with Aquadag, as the collector for the reflected and secondary electrons. Thanks to this improved collector geometry, the saturation of the elastically-reflected electrons was much sharper in the retarding-potential curves. In the side wall of the collector, at an angle of 45° from the normal to the target, was an opening 1 mm in diameter. A special detector was inserted through this opening when it was desired to study the diffraction of slow electrons. The determination of the contact potential difference between the cathode of the electron gun and the target, and the separation of the current due to elastically reflected electrons from the current of true secondary electrons and inelastically scattered electrons, were carried out as in Ref. 1. As in the earlier work, the utmost attention was paid to obtaining and maintaining a sufficiently high vacuum in the tube (the total pressure in the tubes was of the order of 10^{-9} mm Hg, and was 10^{-10} mm or less in those parts of the apparatus where adsorption took place) using getters and ionization pumps.[3] In every case, measurements were made only after the high vacuum had been attained, and immediately after a prolonged high-temperature bake-out (about 2800° K) of the target by electron bombardment, or after the deposition of the material to be investigated.

REFLECTION OF SLOW ELECTRONS FROM MONOCRYSTALLINE TUNGSTEN

The measurements were carried out on a single crystal of tungsten which was at our disposal, consisting of a plate 7 × 7 × 2 mm in size, cut parallel to the (111) plane and chemically etched to remove the layer of material deformed by the mechanical treatment.

The curves of reflection coefficient R and secondary emission δ as a function of the primary electron energy V_p, measured immediately after heating the tungsten single crystal in high vacuum, are shown in Fig. 1 (curves 1). From a comparison of these curves with the data previously obtained using rolled tungsten ribbon,[1] it could be seen that, except for a small difference in absolute magnitude, the curves were quite similar. Consequently, either the rolled tungsten ribbon consists of a uniformly oriented array of crystallites — a widely accepted view at the present time[4] — and is therefore similar to the monocrystal, or else the polycrystallinity of the target plays no role in increasing the reflectivity at higher electron energies. In any case, the single-crystal experiments have not removed the anomaly in the behavior of the reflection coefficient, namely its increase with increasing electron energy.

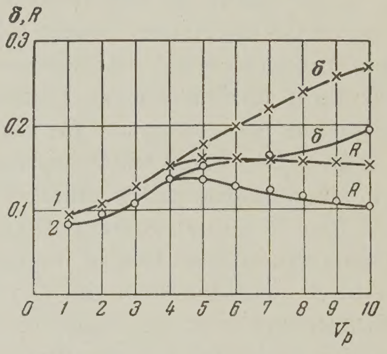

FIG. 1. Variation of R and δ with energy of the primary electrons. 1 – monocrystalline tungsten. 2 – the same covered with a layer of evaporated tungsten.

We have also carried out experiments in which a layer of tungsten was evaporated onto the monocrystalline tungsten target. Such experiments were essential, in view of the fact that in many cases a layer deposited in vacuum has a higher purity than one which has been merely heated, even to the extremely high temperature of the target bake-out. The results obtained with a layer of tungsten evaporated onto the tungsten single crystal are shown in curves 2 of Fig. 1. Deposition of the tungsten has somewhat altered the absolute values of R and δ and has produced a slight falling-off of the curve for R, above $V_p = 5$ ev. Nevertheless, the anomalous behavior in the low-energy region is still there. Hence the increase in reflectivity with increasing energy can hardly be due to contamination of the target surface.

On the basis of MacColl's work[5] and the comments of Herring and Nichols on it (Sec. 28 of Ref. 4) it is possible to attempt a different explanation of these experimental results. MacColl's treatment differed from the earlier work[6,7] by taking into account the periodic nature of the potential within the metal. This led to the prediction of sharp maxima in the curve $R = f(V_p)$, at which R reaches 100%. The effect of inelastic collisions between the electrons inside the metal is to increase the width and greatly decrease the height of these maxima. A distribution in the orientation of individual crystallites, roughness of the sample surface, or a spread in the energies or angles of incidence of the primary electrons, would have the same effect. Consequently it is conceivable that for some metals, because of a particular structure of their internal potential, the reflection coefficient might increase with increasing V_p in a limited region. The idea that the reflection of slow electrons is affected by the potential energy structure of tungsten, in particular, is supported by the correlation between the decreased density of electronic energy states in tungsten and the increased reflection of electrons in the low-energy region. A similar correlation has been observed for copper.[9] The deposition of an active layer which radically alters the profile of the potential at the metal-vacuum boundary can lead to an increased reflection at the surface. This reflection will be of the normal type (a decrease of R with increase of V_p) such as occurs when thin layers of Ba, BaO, or Ba-O (see below) are deposited on tungsten.[1]

REFLECTION OF SLOW ELECTRONS FROM MONOCRYSTALLINE TUNGSTEN COVERED WITH THIN FILMS

To study the potential barrier at the boundary between vacuum and a metal covered with a thin active film, we investigated the reflection of slow electrons from the surface of monocrystalline tungsten covered with a film which reduced the work function. First of all, the experiments with a thin layer of barium atoms[1] were repeated, using the single crystal. This was necessary, since the results previously obtained with a rolled tungsten ribbon coated with barium might have been affected (even if only partially) by the change in the retarding field due to the contact potential of the target after depositing the film. Measurements with a barium film on the tungsten single crystal confirmed our earlier results (see Ref. 1, Fig. 5). Hence the increase in reflectivity as the surface is progressively covered is due to a change in the shape of the potential barrier, and not to a change in electron reflection due to the retarding field.

Layers of the dipolar molecule BaO, which greatly reduce the work function, are of particular interest to the study of cathode electronics. The properties of such systems have been relatively little studied. In this connection, we have experimentally investigated the reflection of slow electrons from a single crystal of tungsten covered with thin layers of BaO molecules. The source of the barium oxide was a spiral of pure platinum covered with barium carbonate, which was heated to convert it to BaO. The amount of barium oxide deposited on the tungsten could be determined from the shift in contact potential, using the data of Gavriliuk.[10] Although it has been stated that the barium oxide deposited from a platinum evaporator is pure,[11] it was considered essential in our particular case to test the evaporation products, and especially to confirm the absence of any appreciable amount of Ba, with a mass spectrometer. These tests* proved that the products of evaporation from our source were in fact pure barium oxide.

Figure 2 shows the results of measuring the coefficient of elastic reflection for the system BaO on monocrystalline tungsten. The shifts in contact potential are indicated in the figure caption. These results depend upon the fraction of the surface covered, which was of the order of one monatomic layer or less. The results obtained with a thick layer of barium oxide will be reported in a separate paper. When barium oxide is deposited, there is a sharp increase in reflectivity in the low-energy region, and the curve $R = f(V_p)$ resumes the normal shape (reflectivity decreasing with increasing energy). The absolute values of R in the low-energy region are considerably larger for the BaO-W system than for the Ba-W system. The results for the latter system are shown as a dashed line in Fig. 2 (see also Fig. 6 of Ref. 1).

In addition to the study of dipolar BaO layers on the tungsten, we also studied the variation in slow electron reflection caused by the evaporation of barium atoms onto a tungsten surface previously covered with a layer of atomic oxygen. The method of producing a layer of atomic oxygen on tungsten is described in Ref. 1. In the present experiment the deposition of oxygen led to an increase of 1.2 ev in the work function, which is somewhat less than the value 1.7 ev obtained in Ref. 1. A comparison of the slow electron reflection from the

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*All the work of testing the BaO source with the mass spectrometer was carried out by G. Ia. Pikus, to whom the author extends his sincere thanks.

BaO-W surfaces with that from the Ba-O-W system shows that, when approximately one atomic layer of material has been deposited, the temperature coefficients are essentially the same and have the same dependence upon energy. From this we conclude that the two systems are largely similar.

It has been suggested several times in the literature that the reduction in work function when a layer of active metal is deposited corresponds with the formation of a potential maximum close to the surface. Some authors[12] consider that the height of this maximum must not exceed the potential level of the vacuum, while others[13] take a different point of view. A study of the reflection of slow electrons may give some evidence to resolve this question; for this purpose, however, we must investigage the interaction of slow electrons with potential barriers containing a maximum. The existing work on this subject[14,15] must be considered unsatisfactory, since the authors make only the crudest assumptions about the shape of the barrier. As a result they obtain an increase in the reflection coefficient at low energies ($R = 1$ at $V_p = 0$). Some calculations have been carried out to find the reflection coefficient from a potential barrier with a maximum lying below the vacuum potential level, formed by superposing the image potential upon the linear potential of a double layer. The results of this calculation predict a relatively small increase in the reflectivity upon going from the pure metal to the metal with an activating layer on its surface, in contradiction to our experimental results. To obtain a satisfactory agreement between theoretical calculations and experiment, it is apparently necessary to assume that when an active metal layer is deposited upon a metal surface, the potential maximum is higher than the vacuum potential level. Such an assumption, and the height of the maximum required to reconcile the results of theory and experiment, is very likely to provide information on the shape of the barrier for this interesting case, which is of such great importance in the study of cathode electronics. It would be very desirable to have such a calculation made. Our preliminary exploratory calculations have shown that, for a potential maximum about 1 ev above the vacuum potential level, it was possible to have a reflection coefficient of the order of 0.2 to 0.4 — i.e., close to the experimental value — for incident electrons of low energy (0 to 1 ev). The difference between the reflection of slow electrons from Ba-W and BaO-W surfaces, on this assumption, implies a difference in the height or width of the potential maximum. In the case of barium oxide the barrier must be considerably wider, and because of the oxygen layer it must also be considerably higher; this would therefore account for the reflection of a greater fraction of the electrons falling on it.

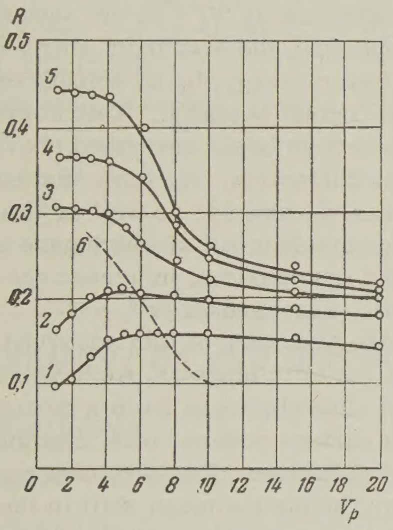

FIG. 2. Variation of R with primary electron energy for BaO-W systems. 1 — $\Delta\varphi = 0$. 2 — $\Delta\varphi = 0.7$ ev. 3 — $\Delta\varphi = 1.7$ ev. 4 — $\Delta\varphi = 2.7$ ev. 5 — $\Delta\varphi = 3.1$ ev. 6 — $\Delta\varphi = 2.4$ ev (Ba-W system).

From our measurements on the systems Ba-W, O-W (Ref. 1), BaO-W and Ba-O-W, it is possible to estimate the effect of slow electron reflection on the size of the constant A in the Richardson-Dushman formula. The reflection of electrons can lead to a reduction in A for such surfaces to 30 — 50% of its theoretical value, 120 amp/cm^2 deg^2. The small values of A quoted in the literature for these systems are apparently connected with other factors, such as the temperature dependence of the work function, non-uniformity of the surface, etc.

DIFFRACTION OF SLOW ELECTRONS FROM MONOCRYSTALLINE TUNGSTEN COVERED WITH A LAYER OF BARIUM

Working with monocrystalline tungsten targets has made it possible for us to observe the diffraction of slow electrons, by using a special collector placed at an angle of 45° to the normal from the target surface. This collector was made positive with respect to the cathode by 3 to 5 volts, so that only the electrons which were elastically scattered from the target would fall on it. The collector was connected to the input of a dc amplifier using a 1E1P tube, giving a sensitivity of 10^{-12} amp/mm. As the potential of the target was varied, the collector current went through a series of maxima (Fig. 3, curve 1).

It is difficult to assign calculated positions to

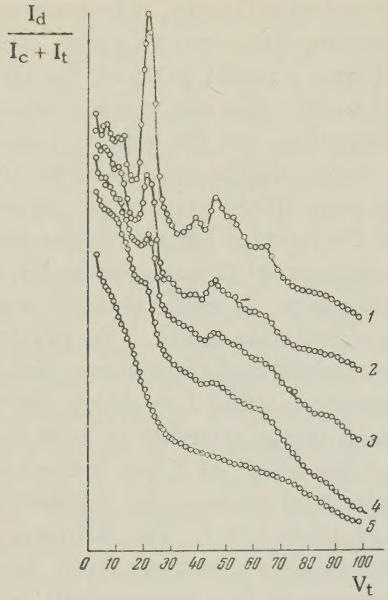

FIG. 3. Variations in diffraction pattern as a layer of Ba is deposited on W. 1 — $\Delta\varphi = 0$ (tungsten). 2 — $\Delta\varphi = 1.1$ ev. 3 — $\Delta\varphi = 1.9$ ev. 4 — $\Delta\varphi = 2.7$ ev. 5 — $\Delta\varphi = 2.0$ ev (thick layer of Ba).

these maxima, since this requires a knowledge of the order of the reflection and the azimuth angle of the collector, i.e., the angle between a particular crystallographic axis in the target and the projection of the line joining the target to the collector. Furthermore, the problem is complicated by the variation of refractive index with energy for electrons in this velocity range, and by the appearance of a large number of supplementary maxima.[16]

Leaving this calculation for the future, we have in the meantime followed a different route and have studied the effect of foreign material deposits on the intensities of the diffraction maxima. In the experimental and theoretical study of slow electron diffraction it is very important to know how many atomic layers are involved. An answer to this question would make it possible to decide between the volume and surface treatments of this effect.[17] However, the number of papers on this subject is not very large. Lashkarev[18] has estimated the number of effective layers from the width of the diffraction maxima. Farnsworth[19] used for this purpose a foreign material (Ag) deposited on monocrystalline copper and gold, and watched the disappearance of the substrate pattern and the appearance of the maxima for silver (the layer of silver deposited on the gold crystal was obtained in an oriented state). In the present work, however, it would be hard to avoid large errors in determining the amount of material deposited, since the method of control is inaccurate. The rate is determined by weighing the amount deposited over a period of a number of hours.

As a matter of interest, we have estimated the number of layers of Ba required to destroy the substrate diffraction pattern, determining the amount of barium deposited from the contact potential change, using Gavriliuk's measurements.[20] Figure 3 shows the diffraction curves obtained by depositing barium onto a cold single crystal of tungsten; for clarity, the curves are displaced slightly along the vertical axis. Curve 4 shows that a barium layer for which $\Delta\varphi = 2.7$ ev almost completely destroys the diffraction pattern. Here the concentration of barium atoms on the surface is approximately monatomic, i.e., 3 to 4×10^{14} atoms/cm^2. It may be concluded from this that the formation of the diffraction pattern is due almost entirely to the uppermost layer of atoms. This argues in favor of Kalashnikov's theory,[17] which treats the diffraction of slow electrons as a purely surface effect. A study of the position of the sharp diffraction maximum at $V_t = 25$ ev shows that as barium is deposited, the maximum shifts in the direction of lower energy, by an amount equal to the change in contact potential. This corresponds with the effects previously described above for slow electron diffraction. We have studied a thick layer of barium (curve 5), as well as thick layers of gold and germanium, all of which gave smooth curves without any maxima, in agreement with our previous work (see Lashkarev[18], p 93).

To study the structure of thin adsorbed layers of barium on metallic tungsten, we have studied the diffraction of slow electrons from a monocrystalline tungsten surface covered with less than a monatomic layer of barium. Two different methods were used for finding the mean shift in the work function $\Delta\varphi$: one measurement was made after the barium had been evaporated onto a clean, cold substrate; and a second was made after the same value of $\Delta\varphi$ was attained by the gradual evaporation of a thick layer of barium when the substrate was heated. Experiment showed that the diffraction patterns obtained from these two surfaces were different. Invariably the surface obtained by evaporating barium off a heated target gave the more sharply defined maxima. In explaining these results it must be kept in mind that when heat is applied to the uniform layer of barium obtained by deposition onto a cold surface, it may result in the formation of an ordered array of barium atoms, and also in the formation of isolated patches of barium. In the latter case, part of the tungsten surface becomes uncovered, resulting in the appearance of its characteristic diffraction maxima.

In general the work of a surface is determined experimentally by the use of electrons with a phase wavelength of the order of a few tens of angstroms, while at the same time diffraction patterns are measured by using electrons with wavelengths of only a few angstroms. Because of this, surfaces with the same average work function may give different diffraction patterns, depending upon their degree of homogeneity. Hence it must be concluded that the change $\Delta\varphi$ in work function is not by any means a single-valued function of the state of a composite monatomic surface.

In conclusion, I wish to express my thanks to Prof. N. D. Morgulis for suggesting the subject, and for his constant interest in the work.

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Translated by D. C. West
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SOVIET PHYSICS JETP VOLUME 34(7), NUMBER 1 JULY, 1958

MEASUREMENT OF THE SURFACE IMPEDANCE OF SUPERCONDUCTORS AT 9,400 MCS

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The surface impedance of several superconductors has been measured at a frequency of 9400 Mcs. The temperature dependence of the surface impedance of thin films of Sn and Hg has been studied. The penetration depth was investigated for these films and found to be in accord with the critical magnetic field measurements. The effective conductivity of the films in the superconducting state increases as the temperature is reduced. An investigation was also made of the relation between the impedance of thin Sn and Hg films and the magnetic field at $T < T_0$. The dependence of penetration depth on field strength is in satisfactory agreement with the Landau-Ginzburg theory.

1. A knowledge of the complex impedance of a superconductor at frequencies on the order of 10^{10} cps makes it possible to find the dielectric permittivity of the metal:

$$\varepsilon = \varepsilon_0 - c^2/\omega^2\delta_0^2. \qquad (1)$$

Determining the penetration depth of the magnetic field δ_0 from static measurements, it is then possible to estimate the value of ϵ_0 due to the presence of bound electrons in the superconductor.[1,2] On the other hand, if $\epsilon_0 \ll c^2/\omega^2\delta_0^2$, it is possible, using these measurements, to find the penetration depth δ_0 of the static magnetic field into the superconductor. Since in a high-frequency field the superconducting electrons do not completely screen the effect of the electromagnetic field on the normal electrons, impedance measurements also furnish data on the normal conductivity of a superconducting current. On the basis of the data reported by Pippard and Simon,[3,4] who measured the total surface resistance of lead at 9400 Mcs, it may be assumed that $\epsilon_0 = 2 \times 10^8$ cgs esu at $T = 3°K$ and that below T_c the ratio of normal-electron conductivity to free path length σ_N/ℓ falls off as the temperature is reduced. Khikin has investigated the surface resistance of thin, superconducting tin layers at the same frequency.[5,6] However, the accuracy of these experiments was not sufficient to justify the conclusion which was reached, namely, that ϵ_0 is large ($\sim 10^9$).

The present work is an extension of experiments in this same subject based on a new and more comprehensive method proposed by Khaikin.

2. The method of investigating the surface impedance $Z = R + iX$ is based on the study of the properties of a hollow, copper, cylindrical resonator, the bottom of which is made from the material being investigated. Since the cavity is excited in the TE_{011} mode, no current flows across the joint between the bottom and the walls of the resonator; thus the nature of the contact between the sample and the copper walls of the resonator is unimportant.

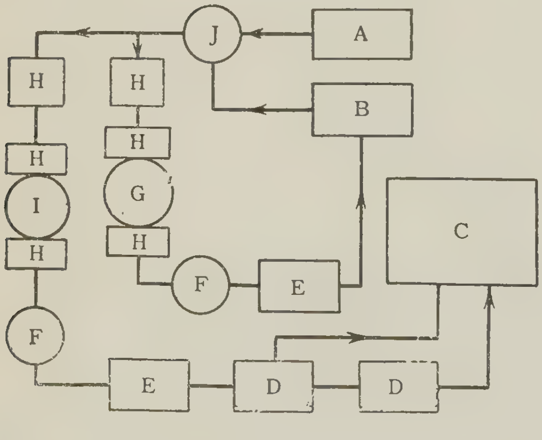

FIG. 1. Block diagram of the system used for measuring the frequency width of the resonator and shift in resonant frequency. A) GSS-6; B) Sawtooth generator; C) EPP-09; D) Outputs I and II; E) Amplifier; F) Detector; G) Standard Cavity Resonator; H) Attenuator; I) Cavity Resonator; J) K-19.

In the experiment, direct measurements are made of the widths of the frequency characteristics and of the change in the resonant frequency of the resonator (Fig. 1). The frequency of the K-19 klystron is modulated periodically by a sawtooth voltage which is applied to the repeller. When the klystron frequency passes through the cavity resonance, a signal is observed at the detector; this signal is proportional to the power transfer coefficient, i.e. it corresponds to the resonator frequency characteristic. The same oscillator excites a standard resonator made of lead and maintained at liquid helium temperature (4.2°K). Since lead is a superconductor below 7°K, this resonator has an extremely narrow pass band (approximately 1 kcs).

The signal from the standard resonator is used as a reference, relative to which the frequency shift of the test cavity is measured. To obtain an absolute frequency scale, a radio-frequency voltage (500 – 5,000 kcs) from a standard signal generator is applied to the repeller of the klystron in addition to the sawtooth voltage. Because the klystron is frequency modulated, its spectrum contains side bands and the signal from the standard resonator has three narrow peaks. The distance between these peaks is determined by the frequency of the standard signal generator and serves as an absolute frequency scale with which the shift of the resonant frequency and the width of the frequency characteristic of the resonator can be measured.

A more precise determination of these quantities is obtained by the use of a special circuit with which the frequency characteristic of the resonator is differentiated twice, producing the following: (a) a voltage proportional to the spacing between the reference point and the point at which the first derivative changes sign, characterizing the change of the resonance frequency of the resonator; (b) a voltage proportional to the distance between the points at which the second derivative of the frequency characteristic change sign, yielding the width of the frequency characteristic of the resonator. Both voltages are recorded by an EPP-09 recording potentiometer. The total accuracy of the measurement circuit is one per cent. To find R and X it is necessary to determine two different quantities: the shift of the resonance frequency of the resonator as the sample makes the transition into the superconducting state and the change in half-width of the resonance curve due to the losses in the sample. The error in the determination of these quantities is about 2 per cent and depends on the proximity to the transition temperature. The probable errors in the determination of ϵ, σ_N/ℓ and σ_{eff}, are shown in the appropriate figures.

The resonator is a copper cylinder with an inner diameter of 43 mm. The resonator is coupled to the K-19 oscillator and the detector through two circular irises 5 mm in diameter cut out of the upper (copper) cover of the resonator; the irises are located symmetrically along a diameter, at a distance of one-half the radius from the center. The irises are terminated by two lengths of circular wave guide with diameter smaller than the critical value; these are used as limiting attenuators between the resonator and the outer coaxial lines. To avoid effects due to motion of the coupling loop on the resonator frequency, the lengths of the attenuators are not changed during the experiment; for this reason the coupling coefficient varies with changing impedance of the sample. The coupling coefficient is never greater than 0.3. The bottom of the resonator is an optically polished plane-parallel slab of smooth quartz, 46 mm in diameter and 3 — 5 mm in thickness, to which the sample is clamped from the inside. The mercury films being investigaged are condensed on the outside of this same slab. The quartz slab is used because the amplitude of the field which acts on the sample is considerably greater than the amplitude of the field acting at the other walls of the resonator because the quartz tends to "concentrate" the field (the dielectric constant of quartz is 3.6); the net result is an increase in the relative fraction of high-frequency resistance of the resonator in the end wall, i.e., the sample being investigated, compared with the rest of the cavity.

The sensitivity is increased by a factor of 3.1 when the quartz slab is 3 mm thick and the resonator length is 39 mm and by a factor of 30 when the thickness of the slab is 5 mm and the length of the resonator 20 mm (in the latter case it is necessary to introduce a correction for the thermal expansion of the quartz in the helium temperature region).

The test cavity is enclosed in a metal jacket in which gaseous helium at a pressure of 20μ is admitted (at $T = 4.2°K$) to provide thermal contact between the resonator and the helium bath. The temperature of the resonator is measured with a thermometer made from a carbon resistor. The voltage drop across the carbon thermometer is recorded by the EPP-09 recording potentiometer together with the resonator characteristics. The error in the temperature determination is less than 0.005°K. In special control experiments with a copper resonator it was established that the high-frequency resistance of the carbon remains constant within the limits of the experimental accuracy. Since the experiments are always carried out with the coupling loop fixed (fixed length for the attenuators between the resonators and the external lines) all changes in the resonator characteristics can be ascribed to changes in the properties of the sample which makes the transition from the normal to the superconducting state. Knowing the change in the width Δf of the frequency characteristic of the resonator produced by losses in the sample and the shift df in the resonance frequency of the resonator, and knowing the geometry of the resonator, it is possible to determine the real part R of the sample impedance and the change in the reactive part dX. To determine the total reactance of the sample one must also know the reactance in the normal state X_n. The total reactance is given by $X = X_n + dX$.

In large samples, where there is an anomalous skin effect, $X_n = \sqrt{3}\,R_n$, where R_n is the resistance of the sample in the normal state, known from experiment. For thin films, the thickness of which is considerably less than the skin depth, $X_n \ll R_n$ to that dX is comparable with R_n; hence, the change in reactance measured in the experiment is essentially the total reactance of the thin layer in the superconducting state.[5]

3. To test the method, the surface impedance of a single-crystal sample of tin of high purity ($\rho_{300}/\rho_{4.2} = 4.10^4$) was measured. The temperature dependence of the dielectric permittivity ϵ of the sample and the ratio of the normal-electron conductivity to free path length σ_N/ℓ, as computed by Abriksov,[1] are shown in Fig. 2. In this same figure are shown the limits of accuracy for the

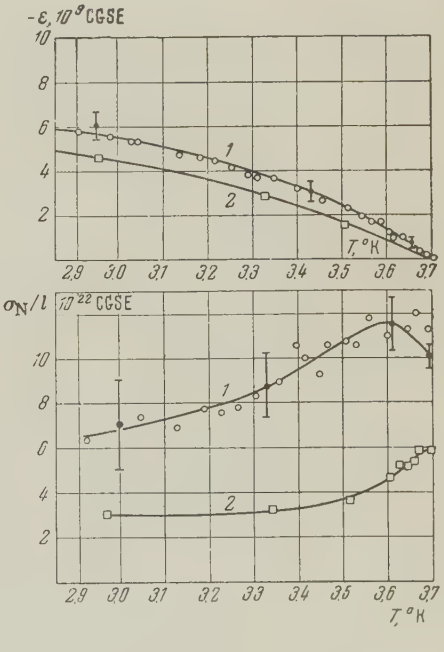

FIG. 2. Dielectric permittivity and the ratio of normal conductivity to free path length in bulk tin. ○— results of the present work; □—results obtained by analysis of the data of Ref. 3 at $\theta = 62°$ K.

measurements of ϵ and σ_N/ℓ at various temperatures. Assuming $\epsilon_0 \ll c^2/\omega^2\delta_0^2$, we can estimate the depth of penetration of the magnetic field into the superconductor δ_0. In the present experiments this quantity obeys the well-known relation between static depth of penetration of the magnetic field and temperature

$$\delta_0 = \delta_{00}/\sqrt{1-(T/T_{\text{к}})^4}. \qquad (2)$$

The value of δ_{00} is in good agreement with the data obtained from dc measurements.[7,8] It follows that ϵ_0 is smaller than the accuracy of the present measurements, i.e., virtually the entire dielectric permittivity of the superconductor is determined by the superconducting electrons; for T = 3°K, $\epsilon_0 < 3.10^8$. Directly below T_c there is a systematic increase in σ_N/ℓ, the magnitude of which is approximately the same as the limits of accuracy of the experiment; as the temperature is reduced beyond this point, σ_N/ℓ is reduced. Similar results are obtained for ϵ and σ_N/ℓ if one analyzes the data of Refs. 3 and 4 using the Abriksov formula.[1] The discrepancy in the absolute values of ϵ and σ_N/ℓ is probably a result of anisotropic effects in tin.

4. Thin mercury films were also investigated. These measurements were made conveniently at liquid helium temperature in the assembled apparatus. The film of mercury was condensed on the optically polished quartz slab which served as the end of the resonator. The evaporator was a copper cup with a nichrome heater in which a drop of mercury was placed. Before evaporation of the film, the heat-exchange gas was completely evacuated from the apparatus by means of an adsorption pump. A power of approximately 0.1 watts was applied to the heater. Under these conditions the temperature of the quartz slab increased at a rate of approximately 0.5°/min. In order to avoid temperature rises of more than 2 — 3 degrees, the precipitation of thick mercury films was carried out in steps. Films with a thickness $d = (5.7 - 36) \times 10^{-6}$ cm were studied. The critical temperature of freshly deposited mercury films was found to be somewhat lower than the critical temperature for bulk mercury (4.15°K), lying in the region 3.9 — 4°K. Impedance measurements were made on freshly deposited and recrystallized mercury films; the latter were annealed at boiling nitrogen temperatures for a period of 18 — 20 hours. The critical temperature of the recrystallized films was 4.15 ± 0.01°K.

The thickness of the films was computed by weighing; the film was weighed by collecting the mercury of the film with the blade of a steel knife.

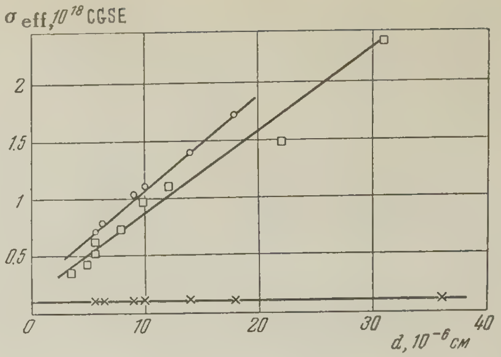

FIG. 3. The effective conductivity of thin films in the normal state as a function of thickness: × — freshly-deposited mercury films, ○ — recrystallized mercury films, □ — tin films.

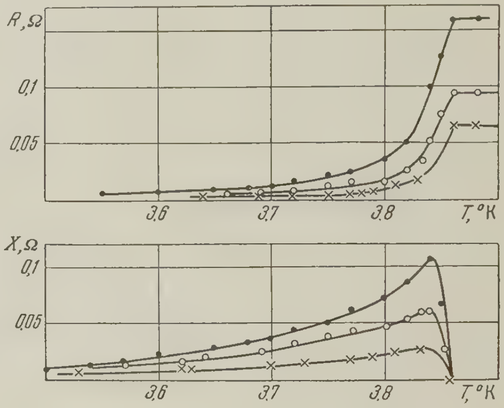

FIG. 4. Resistance and reactance of superconducting thin tin films. The film thickness is given in 10^{-6} cm: ● — 5.35; ○ — 7.7; × — 12.

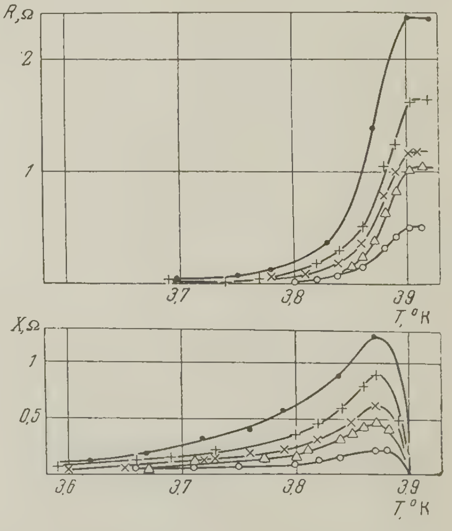

FIG. 5. Resistance and reactance of freshly-deposited superconducting films of mercury. The film thickness is given in 10^{-6} cm: ● — 6.4, + — 10, × — 14, △ — 18, ○ — 36.

The error in the determination of the thickness was less than 2 — 3 per cent. The difference in film thickness between the center and the edge was less than 10 per cent.

Thin tin films were made by deposition of spectrally pure tin (99.918) on a glass plate at room temperature. Film with thicknesses of (3.3 — 33.2) $\times 10^{-6}$ cm were studied. The critical temperature of the tin films (3.8 — 3.9°K) were somewhat above the critical temperature for bulk tin (3.7°K).

The effective conductivity σ_{eff} in the normal state (cf. Fig. 3) was determined from the high-frequency resistance of the mercury and tin films at temperatures above T_c. For tin and recrystallized mercury films σ_{eff} is in good agreement with the theoretical relation $\sigma_{\text{eff}} \sim \sigma_N d/\ell$ (ℓ is the electron free path in the bulk metal) obtained under the assumption that electron diffusion scattering takes place at the boundaries of the film, i.e., that the electron free path in the metal is limited to the thickness. In the unannealed mercury films σ_{eff} was found to be independent of d. This would seem to mean that films condensed at low temperatures have extremely fine crystal structure and are essentially amorphous. The electron free path in this film is considerably less than the thickness of the film. From a comparison of the absolute magnitudes of σ_{eff} in freshly deposited and recrystallized mercury films it may be assumed that $\ell \approx 5 \times 10^{-7}$ cm in the freshly deposited films.

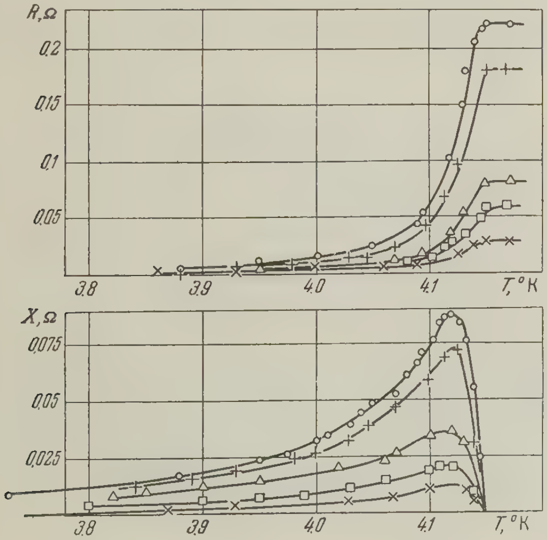

FIG. 6. Resistance and reactance of recrystallized superconducting mercury films. The film thickness is given in 10^{-6} cm: o — 5.7; + — 6.4; △ — 10; □ — 14; × — 18.

The change of surface impedance with temperature as the film goes from the normal to the superconducting state is shown in Figs. 4, 5, and 6. Using the Ginzburg formula,[2] we can compute ϵ and σ_{eff}

$$\varepsilon = -\frac{4\pi}{\omega d}\frac{X}{|Z|^2}, \quad \sigma_{\text{eff}} = \frac{1}{d}\frac{R}{|z|^2}, \tag{3}$$

this formula is valid when $d^2 \ll \delta_{SC}$. This condition is satisfied for films in the normal state and at temperatures 0.3 — 0.4 degrees away from T_c. All calculations were carried out for temperatures close to T_c since Eq. (3) does not apply at lower temperatures; furthermore, there is a significant reduction in the accuracy of the measurements of R and dX.

The results of the calculations of ϵ and σ_{eff} are shown in Figs. 7 — 12. Assuming that the entire dielectric permittivity of the film is due to the superconducting electrons, i.e., $\epsilon_0 \ll c^2/\omega^2\delta_0^2$, it is possible to compute the depth of penetration of the static magnetic field into the superconductor δ_0 and to estimate δ_{00}, using the relation between δ_0 and temperature (2); this relation is found to be in good agreement with the obtained results.

For freshly deposited films $\delta_{00} = (14.5 \pm 1.5) \times 10^{-6}$ cm; for recrystallized films $\delta_{00} = (5.5 \pm 0.5) \times 10^{-6}$ cm. The latter value agrees with the value $(5.3 \pm 0.5) \times 10^{-6}$ cm obtained by Khukhoreva[9] in measurements of δ_{00} in the same films using the critical magnetic field method. For the tin films $\delta_{00} = (6.5 \pm 0.5) \times 10^{-6}$ cm, which is in agreement with the data reported by Zavaritskii.[10] In all films which were investigated σ_{eff} increases as the temperature is reduced in the vicinity of T_c. We may note that an increase of σ_N/ℓ with temperature was noted also in bulk tin (Fig. 2) at temperatures close to T_c. This result is not as yet understood.

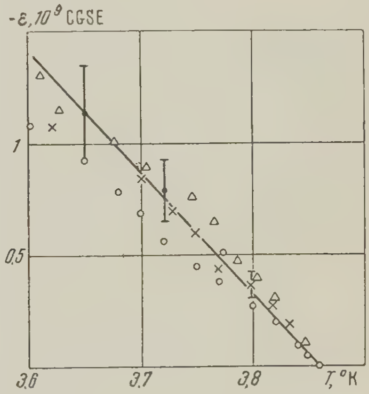

FIG. 7. Dielectric permittivity of superconducting tin films. The quantity d is given in 10^{-6} cm: o — 5.35; × — 7.7; △ — 12; ● — 22.

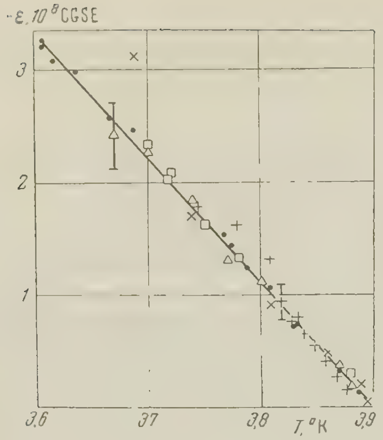

FIG. 8. Dielectric permittivity of freshly-deposited superconducting mercury films. The quantity d is given in 10^{-6} cm: ● — 36; + — 18; □ — 14; △ — 10; × — 7.7.

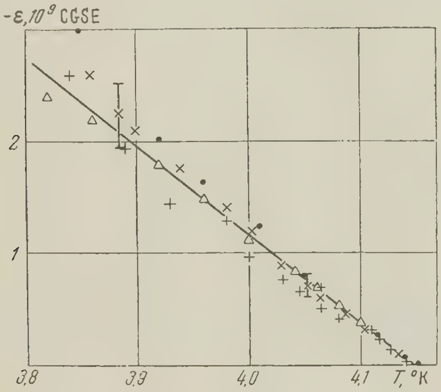

FIG. 9. Dielectric permittivity of recrystallized superconducting mercury films. The quantity d is given in 10^{-6} cm: ● — 18; × — 19; + — 6.6; △ — 10.

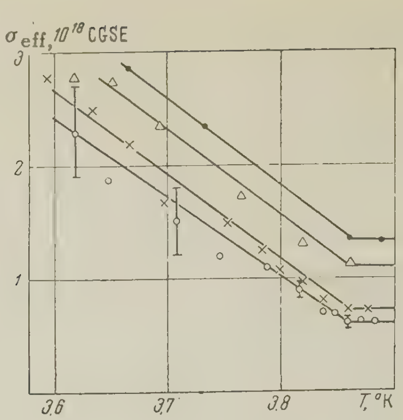

FIG. 10. Effective conductivity of superconducting tin films. The quantity d is given in 10^{-6} cm: ○ — 5.35; × — 7.7; △ — 12; ● — 22.

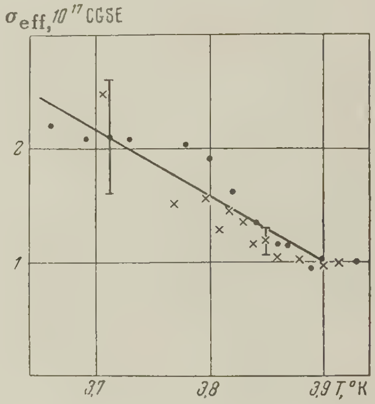

FIG. 11. Effective conductivity of freshly-deposited superconducting mercury films. The quantity d is given in 10^{-6} cm: ● — 36; × — 18.

5. An investigation was also made of the relation between the surface impedance of the superconductor and a fixed magnetic field $H < H_c$. The experiments were carried out with bulk samples and thin films. In order to interpret the results of these experiments it is important that the magnetic field be precisely parallel to the surface of the sample being investigated. In a bulk sample this requirement is automatically satisfied because of the Meissner effect. In thin films the critical magnetic field $H_{c\parallel}$ parallel to the film is several times larger than the field perpendicular to the film $H_{c\perp}$, i.e., when $T < T_c$ the resistance of film is a minimum when the magnetic field is parallel to the plane of the film. This property was used to ensure that the plane of the film was parallel to the magnetic field.

The impedance of the bulk sample of tin remained unchanged, to within an accuracy of 2 per cent, as the field was increased up to $0.91\ H_c$. Consequently, the depth of penetration of the field into the superconductor δ_0 is independent of field (within these same limits). This finding is in agreement with the results obtained by Pippard[3] and is consistent with the Landau-Ginzburg theory,[11] according to which δ_0 in a bulk sample should not change by more than 2 per cent as the magnetic field is varied. In thin samples, the thickness of which is comparable with the depth of penetration at a given temperature, the Landau-Ginzburg theory gives the following dependence of penetration depth on field:

$$\delta_0(H) = \delta_{00} / \sqrt{1-(H/H_c)^2}. \tag{4}$$

The results of the present experiments with mercury films and tin films are in good agreement with this theoretical relation (Fig. 13). A discrepancy between theory and experiment is found only

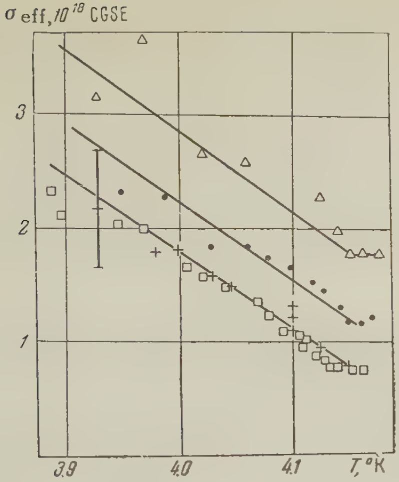

FIG. 12. Effective conductivity of recrystallized superconducting mercury films. The quantity d is given in 10^{-6} cm: △— 18; ●— 14; +—6.4; □—5.3.

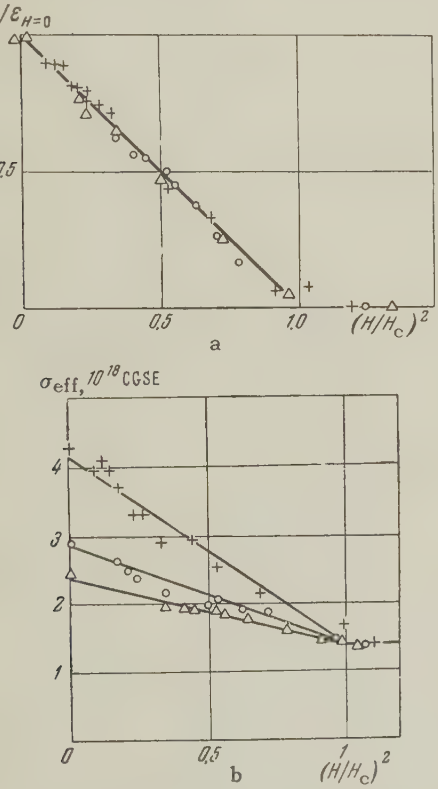

FIG. 13. The relation between dielectric permittivity (a) and effective conductivity (b) and the magnetic field for a tin film of thickness $d = 22 \times 10^{-6}$ cm. △ – T = 3.754°K, ○ – T = 3.724° K, + – T = 3.658° K (T_c = 3.918° K).

in the region close to H_c; this discrepancy is probably due to the fact that in a magnetic field the transition of thin films from the normal state to the superconducting state is not a sharp one.

In all the films which were investigated σ_{eff} is independent of magnetic field in the normal state. In the superconducting state σ_{eff} falls off as the field is increased, approaching the values of σ_{eff} in the normal state at $H = H_c$.

The author is deeply indebted to A. I. Shal' nikov under whose direction the present work was carried out, to M. S. Khaikin for his continued interest and a number of valuable discussions, and to V. L. Ginzburg and B. M. Bolotovskii for a discussion of the results.

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SOVIET PHYSICS JETP VOLUME 34(7), NUMBER 1 JULY, 1958

MAGNETIC SUPERVISCOSITY OF FERRITES

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A study has been made of the magnetic superviscosity curves, at 78°K, of ferrites of the Ni-Zn series, sintered at temperatures from 1200 to 1400°C. The existence of a second maximum of the viscous emf has been observed in a range of fields less than the coercive force, and corresponding to the rapidly rising part of the curve of differential magnetic susceptibility. After the second maximum there is a hyperbolic decrease of the viscous emf curves. On increase of the field, the second maximum merges with the main maximum, and the viscous emf curve becomes approximately exponential. The observed phenomena can be explained within the framework of Néel's theory.

1. OBJECT AND METHOD OF THE RESEARCH

MAGNETIC superviscosity in nickel-zinc ferrites at a temperature of about 80°K was observed by Kuritsyna in 1952.[1] In 1955, Lednev and Telesnin[2] studied the temperature dependence of magnetic superviscosity in Ni-Zn ferrites in the temperature interval from 80°K all the way to the Curie point. Kuritsyna observed two maxima of the superviscosity on the back of the hysteresis loop of high-permeability ferrites; these went over to a single maximum on increase of temperature. In low-permeability ferrites, two maxima were observed at all temperatures, but the duration of the processes was several orders of magnitude smaller.

The object of the present research was to study the actual nature of the process of magnetization change in superviscosity. A series of six specimens was studied. They were of Ni-Zn ferrites of composition 16 mol % NiO, 34 mol % ZnO, the rest Fe_2O_3, sintered at 1200°C (specimen no. 1), 1250° (no. 2), 1270° (no. 3), 1320° (no. 4), 1350° (no. 5), and 1400° (no. 6), for a period of four hours. The temperature of preliminary sintering was 1000°C.

The superviscosity was studied on the back of the hysteresis loop by recording a curve of emf induced in a coil by the viscous change of magnetization that occurred in the specimen after a sudden change of the magnetic field by an amount ΔH. The apparatus used for the recording was a photo-compensated amplifier, F-16, of B. A. Seliber's system, with automatic recording; the response time of its moving system amounted to 0.4 second, and its maximum sensitivity was 2×10^{-8} volt per division. The toroidal specimens were placed in liquid nitrogen.

2. RESULTS OF THE EXPERIMENT

All the specimens exhibited superviscosity. Typical curves of the decrease of viscous emf in specimen 4 are given in Figs. 1a and b. The duration of the process, defined as the time of decrease of the emf curve to 5% of the maximum residue, ranged for various specimens from 33 to 2.5 seconds. In a number of specimens (high-permeability ferrites) a new phenomenon was observed: after the initial residue and the subsequent decrease of the emf, a second maximum was observed; it appeared at some value of the field less than that that corresponding to maximum differential magnetic susceptibility. This maximum appeared 6 to 7 seconds after the change of field; and upon further increase of the field (approaching the value H_c), it occurred earlier and earlier, until finally it merged with the first residue of the recorder; this occurred near the maximum of the differential susceptibility. The time of appearance of the second maximum after the change of field decreased very rapidly, simultaneously with an increase of its amplitude (curves b and c in Fig. 1). The product of the time of appearance of the second maximum and of its amplitude was almost constant, for each of the specimens, and varied little from specimen to specimen. The curves of dependence on the field H of the time of decrease of the viscous emf to 0.05 and 0.1 of the maximum value (the first residue of the recorder pointer) had two maxima; for specimens

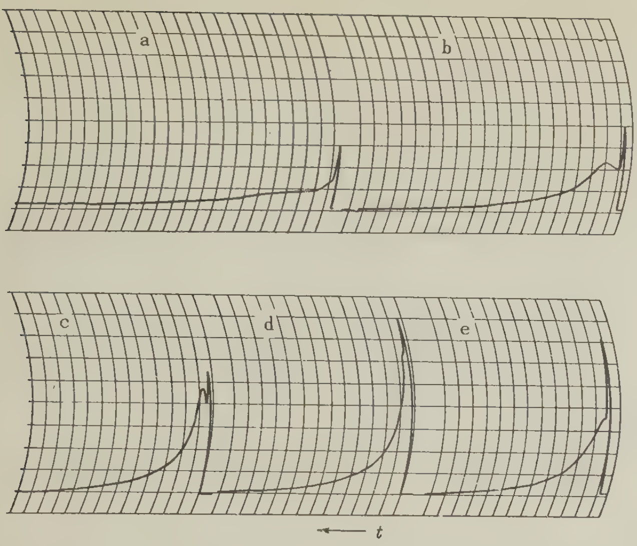

FIG. 1. Curves of the decrease of viscous emf of specimen No. 4. For all curves, $\Delta H = 0.098$ oersted. a — $H = 0.557$; b — $H = 0.588$; c — $H = 0.618$; d — $H = 0.630$; e — $H = 0.704$ oersted.

3, 4, and 6, one lay in the neighborhood of the top bend of the hysteresis curve, the other close to the coercive force. For specimen 5, these maxima were shifted further to the right.

In Fig. 2 are shown the curves of the time τ of occurrence of the second maximum, its amplitude A, the viscosity curves on the basis of a decrease of the emf to 0.1 and to 0.05, the χ_d curve, and the back of the hysteresis loop for specimen 4. In fields corresponding to the maximum of the differential magnetic susceptibility, the second maximum of the viscous emf vanished, and at high fields it did not appear. The curve of the viscous emf after the second maximum is a hyperbola. After the disappearance of the second maximum, the emf curve became transformed to an exponential, $E = E_0 \exp(-t/\tau)$; the value of E_0 was found by extrapolation of the curve to its intersection with the axis of ordinates. The values of the relaxation time ranged between 2 and 3 seconds.

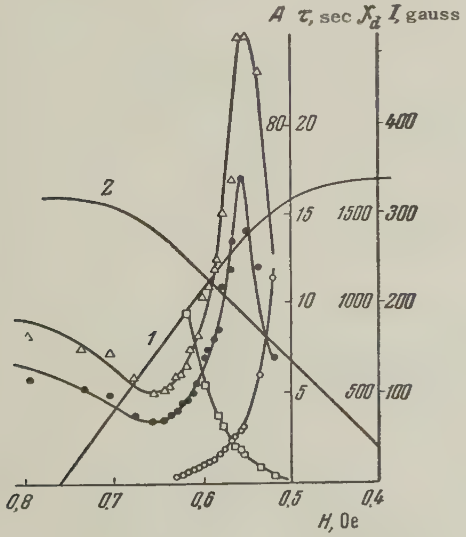

FIG. 2. 1, the back of the hysteresis loop, I; 2, curve of the differential susceptibility, χ_d; Δ and ● — curves of viscosity, $\tau_{0.05}$ and $\tau_{0.01}$; ○ — curve of the time of appearance of the second maximum of τ; □ — amplitudes of the second maximum A for specimen 4.

At small values of the magnetic field, corresponding to the beginning of the rise in the differential susceptibility curve, the second maximum either does not yet exist or is still very small (curve a, Fig. 1); the recorded curve of the decrease of emf is nearly exponential, with the exception of the very first part. Upon further increase of the field, the second maximum appears, then gradually increases and approaches the initial residue (Fig. 1, c and d). The curve of emf decrease after the second maximum is a hyperbola (Fig. 1 c, d, e). The hyperbolic character of the curve is retained until the merging of the two maxima; after this the curve gradually changes to an exponential (Fig. 3, a), again with the exception of the very beginning of the curve.

An interesting matter is the influence of the size of the jump ΔH in magnetic field intensity that is responsible for the viscous change of magnetization. When ΔH exceeds 0.055 oersted and the value of the field intensity H is constant, the time of appearance of the second maximum and its magnitude are almost independent of ΔH; the duration of the decrease of the emf curve is also almost independent of ΔH [the second law of magnetic viscosity (Figs. 3 b and c)]. On decrease of ΔH below 0.055 oersted, the second maximum rapidly decreases and then vanishes (Figs. 3 d and e). For $\Delta H \geq 0.01$ oersted, no viscous process is in general observed, possibly because of the smallness of the viscous emf.

If after commutation from a positive to a negative value of the field, and before switching off of an auxiliary field of less than 0.055 oersted, a delay of about a minute is allowed, the magnitude of the first residue diminishes, and the second maximum vanishes, although the duration of the whole process actually increases (Fig. 3, f). When ΔH exceeds 0.055 oersted, an even larger delay after commutation does not decrease the second maximum and lowers the first residue hardly at all. In Fig. 3 g is given the curve of emf decrease for a five-minute delay after commutation. Figure 4 shows, for specimen 4, the curves of dependence of the viscous emf on the reciprocal of the duration

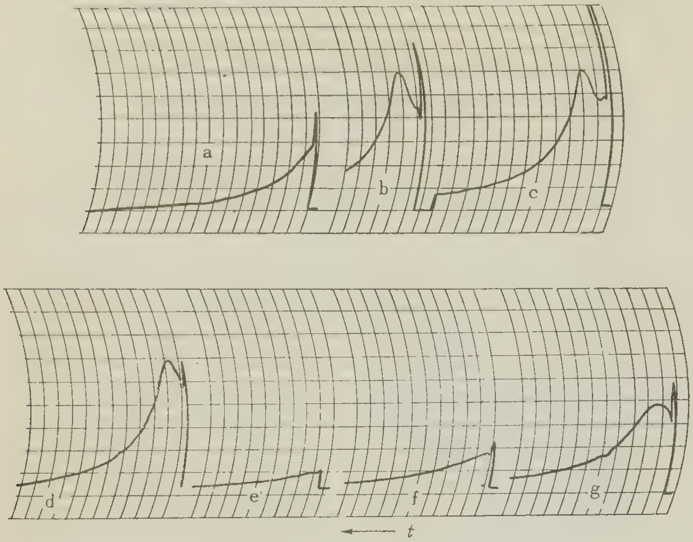

FIG. 3. Curves of the decrease of viscous emf when there is a time delay between commutation of the basic field H and removal of the auxiliary field ΔH. a — H = 0.857 oersted, ΔH = 0.098 oersted, Δt → 0; b — H = 0.566, ΔH = 0.098, Δt → 3 min; c — H = 0.566, ΔH = 0.098, Δt → 0; d — H = 0.566, ΔH = 0.042, Δt → 0; e — H = 0.597, ΔH = 0.024, Δt → 0; f — H = 0.566, ΔH = 0.042, Δt = 5 min; g — H = 0.566, ΔH = 0.098, Δt = 5 min.

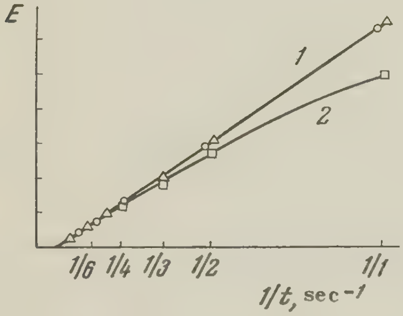

FIG. 4. Dependence of the viscous emf of specimen 4 on the reciprocal of the duration of the process. ○ — H = 0.630; △ — H = 0.636; □ — H = 0.643 oersted.

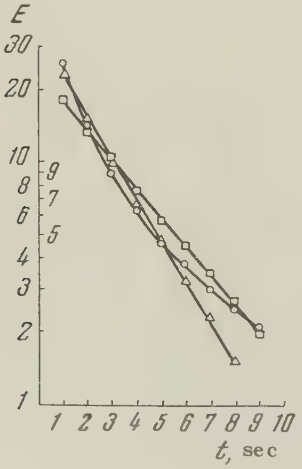

FIG. 5. Dependence of the logarithm of the viscous emf on the duration of the process. ○ — H = 0.643; △ — H = 0.704; □ — H = 0.857 oersted.

of the viscous process. As is evident, at fields of 0.63 and 0.636 oersted the experimental points lie exactly on a single straight line, which corresponds to a hyperbolic dependence $E = (\alpha/t) + b$ (curve 1). On increase of the field to 0.643 oersted, the hyperbolic dependence breaks down (curve 2). On further increase of the magnitude of the field to 0.857 oersted, as is evident from Fig. 5, the dependence of E on t becomes exponential.

3. DISCUSSION OF RESULTS

The observed phenomena can be interpreted within the framework of Néel's theory[3] regarding the existence of two types of magnetic viscosity: reversible and irreversible. According to Néel, reversible viscosity is characterized by the presence of a field H_r opposing the change of magnetization, and which in the case of 180° walls becomes equal to zero at sufficiently large displacement of the wall.

Irreversible viscosity is characterized by an opposing field

$$H_1(t) = S(Q + \log t),$$

where S and Q depend only slightly on log t; they may be considered constant over the whole range of the hysteresis loop.

The viscous change of magnetization is

$$I_1(t) = \chi_d H_1(t) = \chi_d SQ + \chi_d S \log t.$$

At a given point of the hysteresis loop, $\chi_d SQ$ may be set equal to a constant; then

$$I_1(t) = a \log t + \text{const}.$$

This gives a hyperbolic dependence of the viscous emf on time, which was observed in our experiments after appearance of the second maximum

(Fig. 4). The decrease of viscous emf after the first residue and its subsequent increase, giving the second maximum, can be explained by the decrease of the opposing field of reversible viscosity, which occurs in our specimens in the course of a few seconds. The later process is explained on the basis of the process of irreversible viscosity.

The physical mechanism of the process under consideration can be described as follows. On passage to a point that lies near the maximum of the differential susceptibility, some of the domains undergo magnetization reversal; this process proceeds with a speed determined by the diffusion of impurities (reversible viscosity). Under the influence of the reversed domains, other domains, for which there was previously not enough energy for magnetization reversal, begin to reverse. These are "captured" domains, in the terminology of Street and Woolley.[4] This viscous process is determined by energy fluctuation, which gives irreversible viscosity. With approach to the maximum of the differential susceptibility, the number of such "captured" domains becomes smaller and smaller, and a larger and larger number of domains reverse by the reversible process; this leads to coalescence of the maxima, and the change of emf becomes exponential.

An important fact is that after repeated immersion of the specimens in liquid nitrogen, the second maximum decreased considerably, and the whole process described became almost imperceptible. This is attributable to a kind of hardening of the specimen upon repeated violent cooling, which was observed previously by one of us and Lednev.[2] Through such hardening, internal stresses appear in the material, and the energy fluctuations that occur are no longer sufficient for magnetization reversal of "captured" domains.

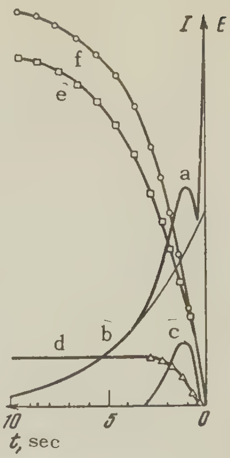

FIG. 6. Analysis of the experimental curve of decrease of viscous emf into two curves, and curves of magnetization obtained by graphical integration. H = 0.60 oersted. a — experimental curve; b — exponential component; c — second component; d — magnetization determined by the second component; e — magnetization determined by the exponential component; f — total magnetization.

The disappearance of the second maximum upon continued exposure to the magnetic field after commutation can be explained by the fact that the field of reversible viscosity is essentially created on commutation of the field, but not on switching off of a field ΔH. Consequently, if we allow a sufficiently long delay after commutation, the impurity atoms responsible for this form of viscosity have time to diffuse into positions determined by the new position of the walls; and a change of field by a small amount ΔH, less than 0.055 oersted, gives only elastic wall displacements, not accompanied by diffusion of atoms. If the magnitude of the change ΔH exceeds 0.055 oersted, for example if it is of order 0.1 oersted, then this process is accompanied by diffusion of impurity atoms; and, independently of the delay time after commutation, the second maximum appears, but the time of its appearance is earlier — about two seconds.

In Fig. 6 the experimental viscous emf curve a is represented as the sum of two emfs — a fast one that decreases according to an exponential law (curve b), and another that first rises and then falls (curve c). The change of magnetization with time is shown in curves d, e, f. Curve e is the magnetization whose change gives the exponentially decreasing emf. Curve d corresponds to curve c, and curve f shows the complete change of magnetization.

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SOVIET PHYSICS JETP VOLUME 34 (7), NUMBER 1 JULY, 1958

LOW ENERGY GAMMA TRANSITIONS IN Tm^{169}

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A magnetic β-spectrometer was used to study the electron spectrum of Yb^{169} in the energy range from 3 — 70 kev. The conversion lines which were found have the following relative intensities: (a) for the 8.32 kev γ-transition, $M_I : M_{II} : M_{III} : M_{IV} = 1 : 0.42 : 0.48 : 0.10$; (b) for the 20.65 kev γ-transition, $L_I : L_{II} : L_{III} = 1 : 0.14 : 0.03$; $M_I : M_{II-V} = 1 : 0.2$. The multipolarity of the 8.32 kev transition is shown to be M1 + E2 (< 3%), while the 20.65 kev transition is M1.

In a series of papers[1-5] on the structure of the energy levels of Tm^{169} it was shown that the experimental spectrum essentially coincides with the theoretical rotation spectrum for nuclei with odd A and spin $\frac{1}{2}$.[6] However, the data[3,5,7] for the low energy γ-transitions ~ 8 and 21 kev, which are difficult to study, were not sufficiently reliable. The purpose of the present work is to investigate these transitions.

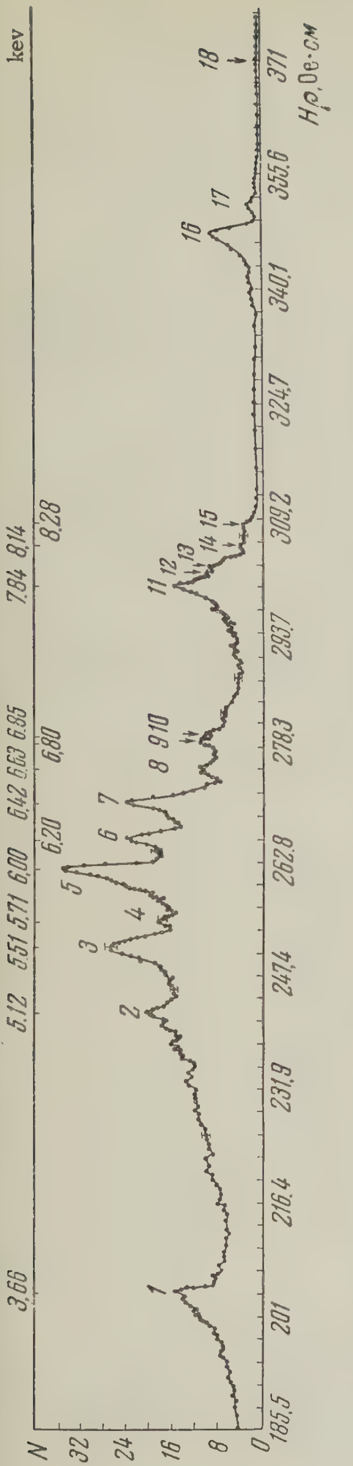
The radioactive source used in our work was Yb^{169}, which decays by K-capture with half-life $T_{\frac{1}{2}} = 30.6$ days,[8] and which was produced by slow neutron irradiation of a natural mixture of isotopes of ytterbium in the Physics-Engineering Reactor.[9] The study of the electron spectrum of Yb^{169} was done with a magnetic beta-spectrometer having $\pi\sqrt{2}$ focusing ($r_0 = 27$ cm). Thin celluloid film was used for the window of the Geiger counter to transmit electrons with energy ~ 2 kev, and for the source backing. The 2 × 25 mm source was prepared by evaporation from an aqueous solution of ytterbium chloride. Aquadag was used to make it a conductor. The spectrometer was calibrated using the conversion lines of Np^{237}. The electron spectrum was studied over the energy interval from 3 to 70 kev. The figure shows the low energy region from 3 to 13 kev. No electron lines were found which did not belong to Tm^{169}. The half width of the conversion lines was ~ 0.7%.

Because of insufficient resolution the Auger spectrum produces a continuous background of electrons. The Auger peaks 2, 3, 4, 8 were detected in the 3 — 8 kev range. From Table I, we see that the theoretical and experimental energy values coincide for these Auger lines.

Let us consider the relative intensity of some of the Auger transitions. We know that in the decay of Yb^{169} a considerable fraction of the γ-transitions are magnetic and electric dipole transitions, which are more strongly converted in the L_I subshell than in the other L subshells. We should therefore expect that the $L_I \rightarrow M_I M_I$ will be the most intense of all the Auger transitions of this series. If we look at the figure, we see that the Auger transition $L_I \rightarrow M_I M_I$ (peak 3) is actually considerably more intense than the next transition $L_I - M_I M_{II}$ (peak 4) and the transitions $L_{II} \rightarrow M_I M_{II}$; $L_{III} \rightarrow M_{II} M_{IV}$ (peak 2) etc. Consequently we should suppose that the Auger lines which coincide with the conversion lines (5, 6, 7 etc.) have still lower intensity and do not produce any essential distortion of the shape of those lines. The interpretation of the Auger lines and conversion electrons is given in Table I.

Conversion of 8.32 kev γ-rays is observed on all the M- and N-subshells. The intensity of the conversion lines (cf. Table I) was computed taking into account the possible contribution from Auger electrons. By comparing the experimental and theoretical values given in Table II for the relative probability of conversion of this γ-ray on the M-subshells (for M1 and E2 transitions), we may conclude that this γ-radiation is a mixture of magnetic dipole and electric quadrupole radiation. Since the absolute conversion coefficient for magnetic dipole radiation is much smaller than for electric quadrupole radiation (from a rough estimate, $\alpha_2/\beta_1 \gg 10$), apparently the admixture of E2 amounts to at most a few percent (< 3).

Conversion of the 20.65 kev γ-rays is observed on the L-, M- and N-subshells. The conversion lines L_{III}; M_I; M_{II}; N; O (#19, 20, 21, 22, 23 in



Low energy region of the Yb^{169} electron spectrum. The ordinates give the number of counts, divided by 64. The numbers on the curve mark the electron lines, whose characteristics are given in Table I.

TABLE I

Number of electron line	Energy of conversion electrons, kev	Shell in which conversion occurs	Energy of γ-transition kev	Intensity in arbitrary units	Multipolarity
1	3,72	K	63.12±0,05		
5	6.00	M_I	8.31	12.00±1.0	
6	6.20	M_{II}	8.29	5.00±0.50	
7	6.42	M_{III}	8.31	5.80±0.75	
9	6.80	M_{IV}	8.32	0.66±0.20	
10	6.85	M_V	8.32	0.54±0.15	
11	7.84	N_I	8.31	5.58±0.40	
12	7.95	N_{II}	8,33	2.96±0.23	
13	7.99	N_{III}	8.33		
14	8.14	$N_{IV,V}$	8.32	0.51±0.13	
15	8.29	O	8.34	0.35±0.12	
		Average	8.32±0,05		M1+E2(<3%)
16	10.51	L_I	20.65	3.56±0.18	
17	10.00	L_{II}	20.63	0.50±0.11	
18	11.94	L_{III}	20.60	0.11±0.03	
19	18.35	M_I	20.66	0.29±0.06	
20	18.53	M_{II}	20.62	0.07±0.03	
21	18.86	M_{III-V}	20.65		
22	20,19	N_{I-V}	20.66	0.07±0.03	
23*	20.63	O	20.66	0.007±0.004	
		Average	20,65±0.07		M1

Auger lines

Number of line	Transition	Electron energy, kev	
		theor.	exptl.
2	$L_{II} \to M_I M_{II}$; $L_{III} \to M_{II} M_{IV,V}$	5,0—5,2	5.12
3	$L_I \to M_I M_I$	5.51	5.51
4	$L_I \to M_I M_{II}$	5,72	5.72
8	$L_I \to M_I M_V$; $L_I \to M_{IV} M_V$	6.56—6.71	6.63

*Electron lines 19-23 are not shown on the figure.

TABLE II

γ-transition kev	Subshells	Relative conversion probability		
		Experimental data	Theoretical value*	
			M1	E2
8,32	$M_I : M_{II} : M_{III} : M_{IV,V}$	$(1.00 \pm 0.13) : (0.42 \pm 0.13) : (0.48 \pm 0.13) : (0.10 \pm 0.03)$	1 : 0.11 : 0.004 : 0,009	1 : 250 : 500 : 35
20,65	$L_I : L_{II} : L_{III}$	$(1.00 \pm 0.05) : (0.14 \pm 0.03) : (0.03 \pm 0.01)$	1 : 0.10 : 0.03	1 : 250 : 250
	$M_I : M_{II-V}$	$(1.0 \pm 0.2) : (0.2 \pm 0.1)$	1 : 0,12	1 : 1000

*Approximate values from Church and Monahan.[12]

Table I), which are not shown on the figure, were detected using a more intense source. Comparison of experimental and theoretical values for the relative probabilities for conversion on L- and M-subshells (for M1 and E2; cf. Table II), shows that the 20.65 kev γ-transition is a pure M1 (with no E2 admixture).

For the 63.12 kev γ-transition, only the K-conversion line #1 is shown on the figure. The intensity of the line is greatly reduced because of absorption in the source and in the film over the window of the counter. We did not study the conversion of these γ-rays in other shells. Except for those enumerated (8.32, 20.65 and 63.12 kev), no other γ-rays were detected in the energy interval 3 — 70 kev.

During the time of this investigation we learned of the work of Hatch et al.,[10] who also studied the low energy γ-transitions in Tm^{169}. These authors state that the 8.42 and 20.75 kev γ-transitions are magnetic dipole (M1),* whereas according to our data the first transition is M1 + E2 ($< 3\%$), while the second is M1. In the same paper it is shown that the rotational band (for $K = \frac{1}{2}$) has levels with spins of $\frac{1}{2}^+$, $\frac{3}{2}^+$, $\frac{5}{2}^+$, and $\frac{7}{2}^+$, while the level with spin $\frac{9}{2}^+$ is absent. Our measurements are in accord with these data.

Computations by Zaretskii based on the coupling scheme treated in Reference 13 show that the 8.32 and 20.65 kev γ-transitions in Tm^{169} should be mainly magnetic dipole, with a possible small admixture of E2 (a few percent). As one can see, the experimental data are not in contradiction with this result.

In conclusion, we express our gratitude to D. V. Timoshuk for much valuable advice.

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SOVIET PHYSICS JETP VOLUME 34(7), NUMBER 1 JULY, 1958

DENSITY OF He^3 — He^4 *SOLUTIONS*

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The density of He^3 — He^4 solutions at saturated-vapor pressure was determined as a function of the temperature at He^3 concentrations of 10, 20.1, 30.3, 41.2, 49.9, 68.5 and 85.4%. The break in the $\rho(T)$ curve affords an estimate of the λ point temperatures for concentrations of 10, 20.1, 30.3, and 41.2%.

THE usual pycnometer method was used to determine the density of He^3 — He^4 solutions. The pycnometer (see Fig. 1) comprised a small glass bulb 1, which narrows down into a capillary 2 of volume 6.5×10^{-3} cm^3/mm and length 29 mm. A platimun "cup" 3 is fused into the lower half of the bulb. This cup is needed to accelerate the equilibrium between the solution in the bulb and the helium in the bath. The cup has a volume of 0.44 cm^3. Marker 4, etched on the capillary, was used as the reference.

Capillary 2 was checked with mercury for absence of taper, after which the volume of the pycnometer was carefully calibrated (accurate to 1×10^{-4} cm^3) by weighing it empty and filled with water to various levels in the capillary. A correction for the expansion of the glass (~ 0.6%) was calculated from the data by van Agt,[1] extrapolated to the region of helium temperatures. An analogous calculation for platinum (data by Henning[2]) yields an equal correction.

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*A supplementary communication[11] states that the 8.42 kev γ-transition has multipolarity M1 + E2.

When working with solutions there is great danger of a change in the concentration of the liquid due to enrichment of the gas phase by the lighter isotope. To reduce to a minimum the harmful volume occupied by the vapor over the solution, a small valve 5 was provided to close off the pycnometer directly at the inlet. The device makes it also easy to correct for the mass of the vapor, which is difficult to determine under conditions when the vapor is contained in the entire filling tube. The pycnometer is connected with the valve through a thin capillary 6, 0.9 mm in diameter and 12 mm long. The valve is fused into a tube of stainless steel, which serves to fill the instrument with a mixture, The valve is operated from the outside with the aid of a rod. The pycnometer volume, not including the valve, is 1.2785 ± 0.0001 cm^3.

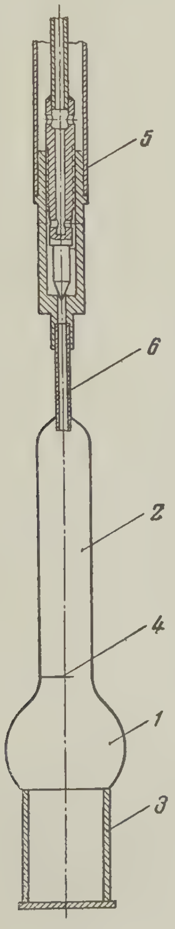

FIG. 1. Pycnometer.

The apparatus used to measure the mass of the gas consisted of a small mercury gas tank 5 liters in volume, a mechanical mercury pump,[3] three manometers to measure the pressures in different parts of the system, and a connecting line. The gas-tank volume was calibrated and was known to within 20 cm^3. The helium-bath temperature was lowered by pumping. The temperature was determined from the vapor tension. The height of the mercury column in the manometer was measured with a cathetometer accurate to 0.1 mm. At low pressures, check readings were made with an oil-filled manometer.

The mixture was made with He^3 of purity not less than 99.98%, as determined with a mass spectrometer.* The solution was prepared for each concentration by mixing known amounts of He^3 and He^4. The concentration was expressed in molar fractions $He^3/(He^3 + He^4)$ and could be determined from the ratio of the He^3 and mixture pressures with a relative error of 0.3%.

The density ρ was measured as follows. Gas was fed to the bulb in small batches from system of total volume V, comprising the mercury pump, gas holder, and manometer. As soon as enough liquid was condensed, the valve was immediately closed and the gas in the line leading from the pycnometer valve to the system (of constant volume V) was pumped back into the system. A KM-5 cathetometer was used to measure the amount of gas remaining after the condensation. In the calculation of the vapor mass, the manometer readings were referred to 0°C and to standard gravity (980.665 cm/sec^2).

During the measurement of the mass of the condensed gas, the room temperature remained constant within 0.5°C. An especially prepared small manometer monitored the absence of vapor in the cold tube over the pycnometer valve during the experiment. The volume of the liquid was measured with the same KM-5 cathetometer, sensitive to 0.01 mm of helium. Actually, since the walls of the helium and air Dewar flasks introduced distortions due to the inhomogeneity of the glass, the accuracy, as determined by control measurements, was somewhat less, but did not drop below 0.05 mm.

The table lists the experimental results for the density ρ of $He^3 - He^4$ solutions, of various concentrations, as functions of the temperature. The

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*The mass-spectrometric analysis of the helium was performed by the laboratory of N. E. Alekseevskii.

Densities of Solutions $He^3 - He^4$

c, %	T, °K	ρ, g/cm^3	c, %	T, °K	ρ, g/cm^3	c, %	T, °K	ρ, g/cm^3
10	1,379	0,13968		1.661	0,12508		3.215	0,10043
	1,535	0,13970		1.764	0,12494		3,333	0.09855
	1,604	0,13982		1,824	0.12492		3.404	0,09770
	1,679	0,14001		1,962	0,12440	68,5	1,309	0,09966
	1.771	0,13982		2.134	0,12388		1.447	0.09948
	1.869	0,14017		2,345	0.12259		1.606	0.09898
	1.996	0,14033		2,682	0,12013		1.651	0,09887*
	2,054	0,14030		3.060	0,11703		1,914	0,09784
	2.095	0,13987		3,257	0,11493		2.284	0.09558
	2,131	0,14005		3,399	0,11307		2.577	0,09320
	2,180	0,13962		3.767	0.10766		2.936	0,08960
	2,194	0,13961	41,2	1,374	0,11769		3,064	0,08763
	2,264	0,13910		1,480	0.11773		3.278	0,08460*
	2,309	0,13908		1,577	0,11771		3,329	0,08347
	2,587	0,13717		1,673	0,11751		3,443	0,08147*
	3,179	0.13203		1,765	0,11732	85,4	1,323	0,08914
	3.592	0,12724		2,058	0.11628		1,355	0,08929
	3.868	0,12312		2,203	0.11552		1,401	0,08914
20.1	1,365	0,13173		2,338	0,11465		1.503	0,08885
	1,447	0,13176		2,630	0,11264		1.646	0,08832
	1,524	0.13177		2,779	0,11157		1,777	0.08766
	1,660	0,13186		2,948	0,10972		2,252	0,08464
	1,762	0,13195		3,122	0,10790		2.486	0.08269
	1,834	0,13205		3,202	0,10698		2,693	0,08016
	1.852	0,13200		3,432	0.10443		2,758	0,07930
	1,885	0,13183		3,572	0.10212		2,787	0,07892*
	1,909	0,13172	49,9	1,333	0,11196		2,871	0,07791
	1,985	0,13143		1.447	0,11180		2,933	0,07667
	2.476	0,12907		1.534	0,11188		3,069	0.07413
	2,705	0,12747		1,669	0,11157	He^3	1,315	0,08111
	3.050	0,12479		1,778	0.11134		1.401	0,08098
	3.365	0,12126		1,883	0,11071		1.600	0,08022
	3.512	0,11941		2.067	0,10983		1,888	0,07868
	3.704	0,11707		2.189	0,10946		2,162	0,07686
30.3	1,339	1,12516		2.334	0,10821		2.360	0.07443
	1,365	0,12512		2.589	0,10665		2.461	0,07347
	1.508	0,12508		2,788	0,10508		2.685	0,07038
	1.579	0.12506		3.009	0,10275			

table contains also the values of the density of pure He^3, measured by the same procedure as used for the solutions. The values obtained are in good agreement with the data of Ref. 4. The temperatures are given in the van Dyke and Duryea 1955 scale. The densities of the solutions were calculated from the formula

$$\rho = \frac{[c\mu_3 + (1 - c)\mu_4]\, pVT_0}{vT\, p_0V_0},$$

where c is the concentration of He^3 in the solution, v the volume of the liquid in the pycnometer, and p and v the pressure and volume of the gas in the system, taken at room temperature T. The subscript 0 denotes the same quantities at standard conditions. The atomic weights of He^3 and He^4 were taken to be $\mu_3 = 3.017$ and $\mu_4 = 4.0039$. Since the volume occupied by the saturated vapor was small, the He^3 content of the vapor did not exceed 0.5 – 0.7% of the condensed He^3 even under the most unfavorable conditions, and the correction for the change in the liquid concentration was therefore disregarded.

The corrections for the mass of the vapor were based on the data on the vapor tensions of solutions,

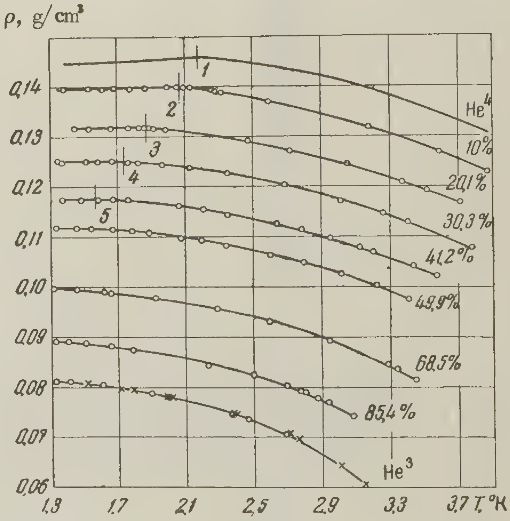

FIG. 2. Dependence of the density of the solutions on the temperature: × – Kerr's data, ○ – results of the present work. The curve for He^4 was plotted from data of Box and Kammerling-Onnes.[7] The vertical bars denote the temperatures of the λ transitions: 1 – 2.17°K (from Ref. 9), 2 – 2.06°K, 3 – 1.87°K, 4 – 1.73°K, 5 – 1.56°K.

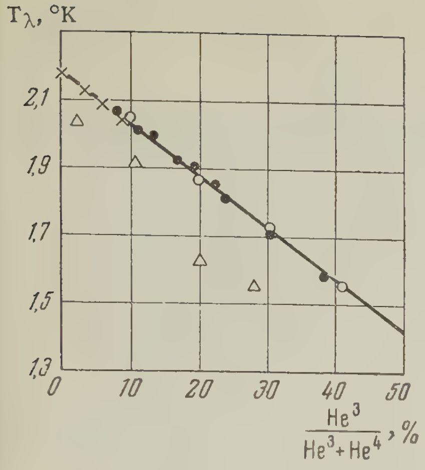

FIG. 3

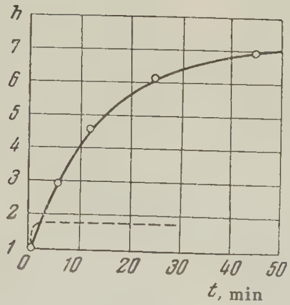

FIG. 4

FIG. 3. Dependence of T_λ on the concentration: × – data of Ref. 9, ● – Esel'son et al.,[8] △ – Abraham et al.,[10] ○ – results of the present work.

FIG. 4. Curves showing establishment of equilibrium. h – height of liquid column in the bulb (in arbitrary units), t – time in minutes. Solid curve – T = 1.852°K, dotted curve – T = 1.885°K.

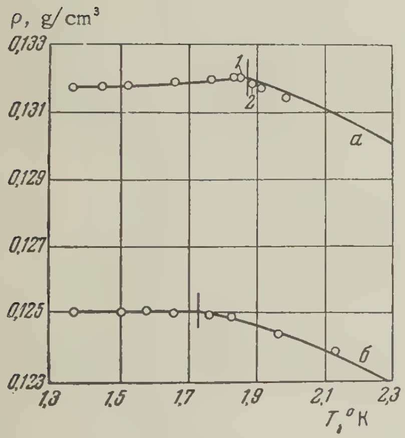

FIG. 5

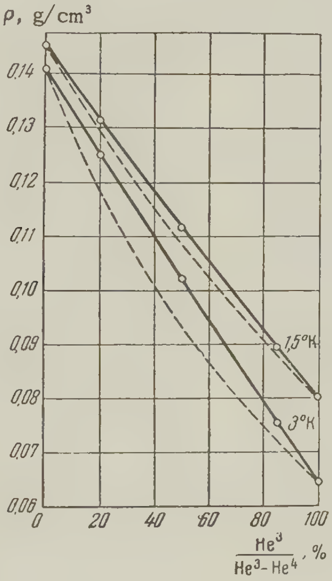

FIG. 6.

FIG. 5. Dependence of ρ(T) in the vicinity of the λ point for concentrations of 20.1 (a) and 30.3% (b).

FIG. 6. Dependence of the density of the solutions on the concentration. Solid curve – experimental, dotted curve – calculated under the assumption that molar volumes are additive.

obtained by Esel'son and Berezniak.[5] The vapor pressure of He^3 was obtained from Ref. 6. The experimental results are plotted in Fig. 2.

Three to five experiments were performed for each concentration, to avoid random errors due to incorrect determination of the mass of the condensed gas. The spread of the points relative to each of the curves for ρ(T) does not exceed 0.2%, although the estimated possible systematic error given is on the order of 1%. It is interesting to note that as the concentration increases, the fairly sharp break at the λ point, characteristic of the ρ(T) curve of He^4, becomes smoothed out. The densities below T_λ become constant within 0.1% for concentrations of 30.3 and 41.2%.

The break in the ρ(T) curve made it possible to determine the temperatures of the λ transition for the 10, 20.1, 30.3, and 41.2% concentrations. The λ temperatures were determined for the last two concentrations from the intersection of the tangents at the bend. The points determined in this manner (Fig. 3) are in good agreement with the results of Ref. 8 to within an experimental accuracy of ± 0.05°K.

Let us note that it takes much longer to establish equilibrium in the solutions at temperatures below T_λ (50 – 60 minutes) than at temperatures above the λ transition, when the measurements can be made in practice within 3 – 5 minutes after the helium-bath temperature is established. Since control measurements with pure He^4 have shown that the equilibrium is reached here very rapidly (3 – 5 minutes) over the entire investigated temperature range from 1.3 to 4°K, we can conclude that as the bath temperature is lowered, the pycnometer becomes unevenly cooled and the normal liquid carries the He^3 towards the colder portion of the apparatus; then, as the concentration becomes equalized, the system acquires equilibrium through a slow diffusion process. Figure 4 shows typical settling curves for one of the experiments. The solid curve was plotted for the point 1 of Fig. 5, and the dotted one for point 2. Comparison of the two curves shows that point 2 lies above T_λ, while point 1 is in the He II region. Thus, the temperature of the λ transition can also be determined from the character of the equilibrium curves.

To verify that the measurements made at temperatures above T_λ were really performed in the equilibrium state, several points (marked in the table by asterisks) were obtained under conditions, in which the required temperature was obtained by placing a heater under the pycnometer bulb. After the bath temperature was fixed, the heater was turned off and the dependence of the position of the level in the capillary on the time was plotted. The points corresponding to equilibrium under these conditions fit the $\rho(T)$ curves quite well.

The experimental values of the densities of the solutions differ somewhat from the theoretical ones, calculated under the assumption that the molar volumes are additive (Fig. 6). For a given concentration, this difference increases with the temperature, reaching, for example, 8.5% at $c = 50\%$ and $T = 3°K$. Such a behavior of the density corresponds to a reduction in the molar volumes of He^3 and He^4 upon dissolution.

In conclusion, I express deep gratitude to V. P. Peshkov for valuable advice and help in this work.

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TEMPERATURE DEPENDENCE OF PARAMAGNETIC RESONANCE ABSORPTION AT CENTIMETER WAVELENGTHS

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Paramagnetic resonance absorption at centimeter wavelengths has been investigated in the temperature region 90 — 500°K in $CrCl_3$, MnF_2, MnO, CuF_2 and $Cr(OH)_3$. It was found that in $CrCl_3$ and CuF_2 the line width is temperature independent while the intensity varies in accordance with the Curie-Weiss law. In MnF_2 and MnO the temperature dependence is more complicated, indicating the existence, in these compounds, of residual antiferromagnetism at temperatures above the Curie point.

THE absorption measurements reported here were carried out in the temperature range 90 — 750°K using the standing-wave method. The details of the method are described in Ref. 1.

The measurements were carried out with a special system, a block diagram of which is shown in Fig. 1. The rf source was a klystron with a stabilized power supply. The klystron power was monitored with a thermistor bridge. The measuring cavity was a shorted rectangular waveguide excited in the TE_{011} mode; a sample of the material investigated was placed at an antinode of the magnetic field.[2] The effect of χ' on the resonance absorption curve was small. To avoid this effect the method described in Ref. 1 was employed. In the measurements of paramagnetic absorption at low temperatures, use was made of a supplementary wave guide system made from a thermally insulating material with an inner conducting surface; this was placed in a Dewar flask with liquid nitrogen or oxygen. The measurements at high temperatures were carried out with a measuring waveguide enclosed in an electric heater, with which the sample could be heated to 500°C. The temperature was measured with a platinum-rhodium thermocouple, the junction of which was attached to the end wall of the waveguide. The measurement waveguides were connected by choke joints between which there were insulators made from thin mica sheets (0.04 mm thick) at a distance of 420 mm from the end of the waveguide. The space between the measuring line and the cavity was covered by a metal jacket which was cooled by circulating water.

Curves showing the quantity χ''_{max} as a function of H_0 were plotted in the temperature range between 90° and 500°K for the following salts: $CrCl_3$, MnF_2, MnO, CuF_2, $Cr(OH)_3$. The measurements were carried out at a frequency ν = 9378 Mcs.

In Fig. 2 are shown the results of an investigation of paramagnetic absorption at different temperatures in $CrCl_3$. In the same figure is shown the absorption strength at maximum χ''_{max} in

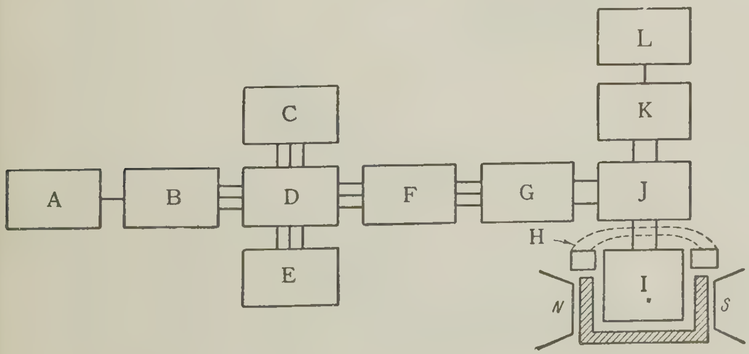

FIG. 1. Block diagram of the klystron oscillator. A) ferromagnetic resonance stabilizer; B) electronic stabilizer; C) power monitor; D) oscillator; E) resonance wave meter; F) variable attentuator; G) choke joint; H) cooling system; I) absorption cell; J) heater; K) detector; L) recorder.

$CrCl_3$ as a function of temperature. These results indicate that at all temperatures the line width of the curve remains constant ($\Delta H \sim 170$ oersted), at a value corresponding to a transverse relaxation time $\tau_2 = h/2\pi g\beta\Delta H = 3.2 \times 10^{-10}$ sec. The line width for $CrCl_3$ is in agreement with the results of earlier measurements.[3,4]

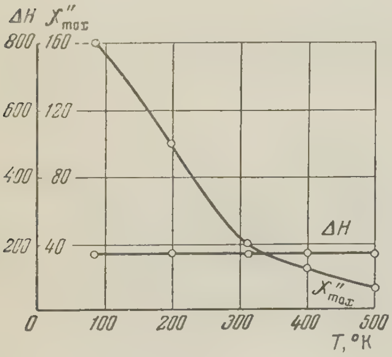

FIG. 2. ΔH and χ''_{max} as functions of temperature in $CrCl_3$; $\lambda = 3.21$ cm.

Kozyrev and the author,[5] on the basis of an investigation of paramagnetic resonance absorption in a number of salts of copper, chromium, and manganese at several frequencies, have established the fact that

$$\frac{\chi''_{max}\Delta H}{\chi_0} = \text{const}, \qquad (1)$$

where χ_0 is the static magnetic susceptibility. The experiments described here indicate that Eq. (1) also applies when the temperature is varied:

$$\frac{\chi''_{max}(T)\,\Delta H(T)}{\chi_0(T)} \approx \text{const} = W. \qquad (2)$$

In $CrCl_3$ this relation is valid over a wide temperature region, as is apparent from Table I.

It is also apparent from the table that the computed value of W remains constant over the entire temperature region which was investigated.

In Fig. 3 are shown the results of measurements of resonance absorption in MnF_2. The antiferromagnetic Curie point in this compound is $\Theta_{anti} = 72°K$. It is evident that the line width ΔH in MnF_2 is reduced by approximately a factor of 2 as the temperature is increased from 90° to 350°K. Table II lists the results of a calculation of W based on Eq. (2).

The temperature dependence of the magnetic susceptibility χ_0 is taken from Refs. 8 and 9.

It is apparent from Table II that in MnF_2 Eq. (2) applies only for temperatures between 300° and 350°K; below 300°K the quantity in question falls off monotonically with diminishing temperature. This effect is undoubtedly due to the effect that the antiferromagnetic properties of MnF_2 have on W above the Curie temperature. As a matter of fact, neutron diffraction experiments in MnF_2 indicate that even at room temperature the scattering of this material is not completely paramagnetic.[7]

A similar pattern for W is found in other antiferromagnetic materials at temperatures above the Curie point. Thus, for example, we have the case of manganese oxide MnO (Fig. 4). In this material the antiferromagnetic transition point (Curie point) $\Theta_{anti} = 116°K$. The results of a calculation of W in MnO_2 using Eq. (2) are shown in Table III. It is apparent from the table that the residual antiferromagnetism continues to have an effect on

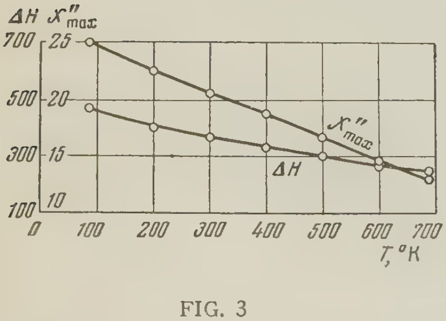

FIG. 3

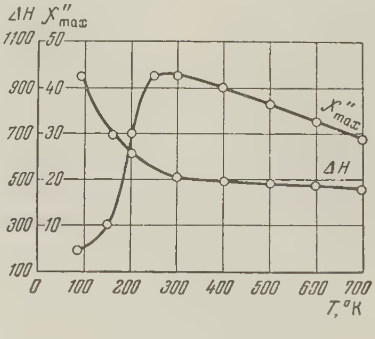

FIG. 4

FIG. 3. ΔH and χ''_{max} as functions of temperature in MnF_2.

FIG. 4. ΔH and χ''_{max} as functions of temperature in MnO.

TABLE I

T, °K	90	200	310	400	500
$\chi_0 \cdot 10^6$	1.72	100	52	34	20
χ''_{max}, arbitrary units	122	75	40	25	15
ΔH, gauss	170	170	170	170	170
W, arbitrary units	124	128	128	124	128

TABLE II

T°, K	90	200	300	350
$\chi_0 \cdot 10^6$	20.0	12.8	9.60	8.7
χ''_{max}, arbitrary units	25.0	23.0	21.0	19.5
ΔH, gauss	465	400	360	342
W, arbitrary units	58.0	72.0	78.0	77.0

TABLE III

$T°$, K	90	200	260	294	400	500	600	700
$\chi_0 \cdot 10^6$	90	81	79	78	65	58	52	45
χ''_{max}, arbitrary units	5	30	37	43	40	36	33	29
ΔH, G	950	616	560	500	495	490	480	470
W, arbitrary units	535	228	265	304	304	304	304	304

W at room temperature. Starting at room temperature, however, and up to 700°K, MnO_2 apparently exhibits paramagnetic properties. Approximately the same behavior was observed in MnS.

Thus, the present experiment on MnF_2, MnO and MnS, as well as neutron diffraction experiments in MnF_2,[9] indicate the presence of residual antiferromagnetism at temperatures above the Curie point.[10]

In order to compute the effect of the antiferromagnetic properties of MnF_2 on the spin-lattice relaxation time close to the Curie point, special experiments were performed to examine paramagnetic fields at T = 77°K. A spin-lattice absorption depending on the external magnetic field was not found in the frequency region 15 — 600 Mcs. It is possible that the spin-lattice relaxation time in MnF_2 is of the same order of magnitude as the spin-spin relaxation time τ_s.

In addition to the three indicated Mn^{++} compounds, we studied $Cr(OH)_3$ and CuF_2. In the first material the line width ΔH falls from a value of approximately 600 oersted at 90°K to approximately 400 oersted at 290°K and remains constant as the temperature is increased beyond this value. The strength of the maximum in the resonance absorption χ''_{max} increases as the temperature is increased, reaching its highest value at room temperature and then falling off slowly to one-third of its value at T = 523°K. Thus, the behavior of $Cr(OH)_3$ is similar to that of MnO. It was not possible to check the constancy of W in $Cr(OH)_3$ because of the lack of data on $\chi_0(T)$. However, the functions $\chi''_{max}(T)$ and $\Delta H(T)$ in $Cr(OH)_3$ would seem to indicate that residual antiferromagnetism in this material can still be found at room temperature.

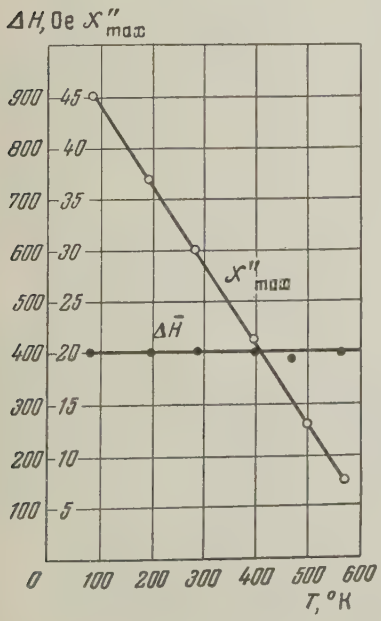

FIG. 5. ΔH and χ''_{max} as functions of temperature in CuF_2.

The results of the measurements in CuF_2 are shown in Figs. 5 and 6. This material is antiferromagnetic at temperatures below 78°K. However,

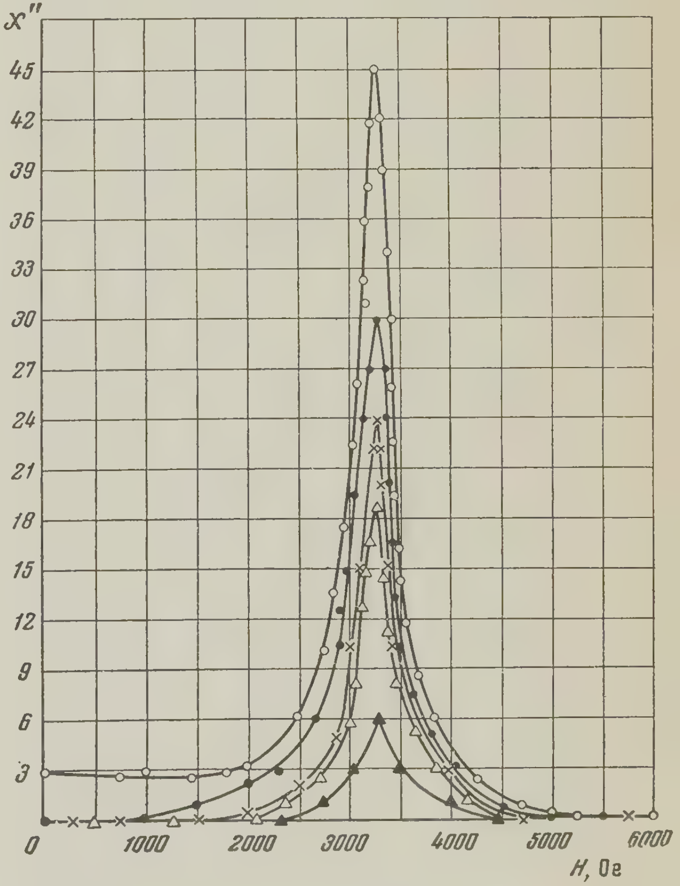

FIG. 6. Resonance absorption curves in CuF_2 at various temperatures: ○ – T = 90°K; ● – T = 290°K; × – T = 403°K; Δ – T = 473°K; ▲ – T = 573°K.

no anomalous behavior above the Curie point, similar to that observed in the other antiferromagnetic materials, was found in this case. The line width for CuF_2 was constant in the temperature region investigated and equal to 400 oersted. The strength of the resonance absorption at the maximum was reduced by a factor of one and one-half as the temperature was increased. Eq. (2) is in good agreement with the results found in CuF_2 over the entire temperature region from 90° to 294°K. Thus the behavior of CuF_2 above the Curie point is that of a typical paramagetic material.

In conclusion we may note that the effective g-values for all materials investigated were close to 2 and temperature independent within the limits of the experimental accuracy. The experimental resonance absorption curves are a better fit to a Lorentzian line-shape than a Gaussian shape.[11]

This work was carried out under the guidance of B. M. Kozyrev.

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CLOUD CHAMBER INVESTIGATION OF THE NUCLEAR-ACTIVE COMPONENT OF AIR SHOWERS

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Nuclear processes induced by particles of extensive air showers were studied by means of a cloud chamber containing seven lead plates. The lateral distribution of nuclear-active particles in extensive showers as well as the nature and the energy spectrum of the particles were found.

THE nuclear-active component of extensive air showers was studied by means of the previously described experimental arrangement[1] consisting of a cloud chamber[2] and a counter telescope. Seven lead plates, 0.5, 1.5, 2, 2, 2.5, 2, and 2.5 cm in thickness (top to bottom respectively) were placed in the cloud chamber 60 × 60 × 30 cm. The effective chamber area for detection of nuclear-active particles, accounting for their angular distribution, was equal to 0.15 m^2. A shower-detecting counter system was used to trigger the hodoscope and the cloud chamber. The hodoscope made it possible to determine the distance of the shower axis from the cloud chamber and the total number of particles in the shower.

A particle was considered nuclear-active if it produced a shower in a lead plate, satisfying at least one of the following requirements:

(1) At least two penetrating particles could be found among the secondaries when the primary was charged, and at least one when the primary particle was neutral. A particle was considered as penetrating if it traversed at least one lead plate without multiplication.

(2) At least one heavily ionizing particle was present among the secondaries.

2175 extensive air showers were recorded during 1189 hours of operation. In these, 30 nuclear-active particles were detected. The number of chance coincidences between a single nuclear-active particle and an extensive shower was calculated and found to be less than 1 during the whole period of operation.

1. LATERAL DISTRIBUTION OF NUCLEAR-ACTIVE PARTICLES

We found first the flux density $\rho_n(r)$ of nuclear-active particles in extensive air showers at various distances from the axis using the well-known formula

$$\rho_n(r) = \frac{N_n(r)}{C(r)\,\sigma\,[1-\exp(-x/\lambda)]},$$

where $C(r)$ is the total number of extensive air showers with axes falling at the distance r from the cloud chamber, $N_n(r)$ is the number of nuclear-active particles at the distance r, σ is the effective area of the chamber, λ is the interaction mean free path of nuclear-active particles (in lead $\lambda = 160$ g/cm^2), and x is the total thickness of lead plates in the chamber ($x = 140$ g/cm^2).

The results for extensive air showers with the mean number of particles $\bar{N} = 2 \times 10^5$ are shown in Fig. 1. The large statistical errors make it impossible to conclude about the exact shape of the lateral distribution function of nuclear-active particles. The results, however, do not contradict a function of the type r^{-n} with n close to 1, which is in agreement with the results of Ref. 3, in which a hodoscope covered with lead was used for the detection of nuclear-active particles.

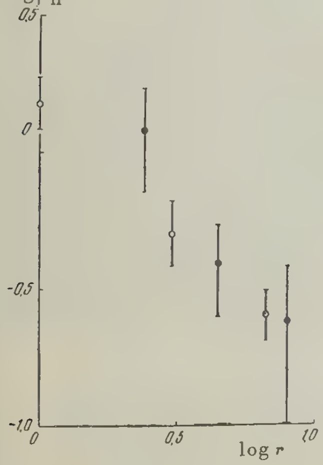

FIG. 1. Lateral distribution of nuclear-active particles.

It should be noted that our data obtained by means of a cloud chamber are in agreement, with respect to the absolute value of the flux of nuclear-active particles, with the results obtained at sea level by means of a hodoscope array.[3] The light circles in Fig. 1 denote the experimental points referring to showers with $\bar{N} = 3 \times 10^5$, given in Ref. 3. Since the identification of nuclear-active particles by means of a cloud chamber is more positive, the agreement in the values for the absolute density of the particle flux proves the correctness of the criterion for the selection of nuclear-active particles assumed in the counter work.

2. ANALYSIS OF NUCLEAR INTERACTIONS OBSERVED IN THE CLOUD CHAMBER

A careful study of the electron-nuclear showers generated by nuclear-active particles showed the existence of two separate groups:

The first group comprised electron-nuclear showers containing dense electronic cascades. Thirteen such showers were detected. In ten of these, penetrating or heavily ionizing particles were present in addition to electrons. An example of such a shower is shown in Fig. 2. In the remaining three events no particles other than electrons could be observed due to high particle density. These three cases were produced by neutral primaries with path length in lead longer than 2 cm. The primaries could not have been identified as photons since the probability that a photon traverses without multiplication more than 4 radiation units is equal to 0.05. Moreover, only 8 photons of a similar energy were detected at all. In the first group of showers penetrating particles, if visible, were going at relatively small angles to the direction of the electronic cascade.

The second group consisted of electron-nuclear showers in which high-energy electron cascades were absent. A rather large number (about 10) of penetrating particles going at widely distributed angles, heavily ionizing particles, and low-energy electrons absorbed in next lead plate, were present in these showers. Eight such cases were recorded. An example of a shower of that type is shown in Fig. 3.

A narrow angular distribution of particles and large energy of π^0 mesons generating the electron-photon cascades are, therefore, the characteristic features of the first group of showers. The showers of the second group are, on the other hand, characterized by a wide lateral distribution of particles and a relatively low energy of π^0 mesons.

It follows from the well-known formula of relativistic angle transformation that as the velocity of the center of mass of colliding particles increases, the spread of produced particles in the laboratory system becomes narrower. It can be assumed therefore that the showers of the second group were induced by collisions with a small velocity of the center-of-mass system. Since, however, these showers consist of large numbers of particles, the primary energy should be large.

On the strength of the above, we assumed that the showers of the first type originated when the primary particle collided with a small number of nucleons in the nucleus, in the main with one only, and the showers of the second group were produced in collisions between the primary and several nucleons of the lead nucleus. In that case, the velocity of the center of mass may be small even for a relatively large energy of the primary because of the large mass of all nucleons involved in the collision.

Two smaller groups of electron-nuclear cascades were observed besides the two main types. The third group consisted of four cases only. The showers were of low energy and the secondaries consisted of a few heavily ionizing particles, a few electrons, and 1 — 2 penetrating particles. The latter might have been also absent. An example of a shower belonging to this group is shown in Fig. 4. Finally, the fourth group of showers, consisting of five events, contained showers originating in the last, 7th, plate. In that case it was impossible to differentiate between penetrating particles and electrons and it was therefore impossible to find any characteristic features which would enable the classification of the event within one of the three groups. Additional criteria (neutral primary, presence of heavily ionizing secondaries) made it possible to ascertain that the showers belonged to the electron-nuclear type. An example of such a shower is shown in Fig. 5. Data on all recorded showers are given in Table I.

3. DETERMINATION OF THE ENERGY OF NUCLEAR-ACTIVE PARTICLES

The energy of nuclear-active particles was determined for the different groups of showers by a method most suitable for the type of event.

We used the method of Ref. 4 for determination of the energy of nuclear-active particles generating the showers of the second group. This method is based on the application of the statistical theory of Fermi[5] to collisions of the primary with several nucleons of the target nucleus and consists in finding the energy of the primary particle and the number of nucleons involved in the collision from the number of penetrating secondaries and the velocity of the center of mass of colliding particles. The first is observed directly, and the second can be found from the angular distribution of secondary particles if we assume that (1) the distribution is isotropic in the center-of-mass system and (2) the velocities of particles in that system are close to c.

Under these assumptions one can obtain easily the relation between the velocity of the center of mass, or the value $\gamma_c = (1 - v^2/c^2)^{-1/2}$ and the

TABLE I. Data on nuclear interactions*

Charged primary				Neutral primary				Charge of primary undetermined			
Case no.	r, m	N	E_k (ev)	Case no.	r, m	N	E_k (ev)	Case no.	r, m	N	E_k (ev)
22.22	0,5	$4\cdot10^4$	$7\cdot10^9$	3.30	3	$1,3\cdot10^4$	$1.5\cdot10^{10}$	18,29	7	$6\cdot10^4$	$1.5\cdot10^{10}$
28.32	2,6	$5,3\cdot10^5$	$3.7\cdot10^{10}$	28,4**	8	$1,5\cdot10^6$	$2.5\cdot10^{10}$	40,16	9	$1,5\cdot10^5$	$6\cdot10^9$
73,39	3,2	$4\cdot10^4$	$2,3\cdot10^{10}$	36,12	3,5	$8\cdot10^4$	$3\cdot10^{10}$	15,49**	0,5	$7.4\cdot10^3$	$2\cdot10^{10}$
57,10	1,4	$1,1\cdot10^4$	10^9	45,10	0,5	$7,4\cdot10^4$	$3\cdot10^{10}$	22,72**	0,5	$2.5\cdot10^5$	$1,5\cdot10^{10}$
60,14	2,7	$3,1\cdot10^4$	$2\cdot10^9$	46,1	3,1	$2\cdot10^5$	$6\cdot10^9$	44,14	4	$1,1\cdot10^5$	$1.2\cdot10^{10}$
78,20	2,8	$1,5\cdot10^5$	$3\cdot10^{10}$	63,7		—	$4\ 3\cdot10^9$	55,1	6,7	$3\cdot10^6$	—
76,51	2,5	$2,4\cdot10^5$	$1,5\cdot10^{10}$	75,4	4,6	$1,9\cdot10^5$	10^9	70,37	5	$2.3\cdot10^5$	$1,5\cdot10^{10}$
82,21**	2,5	$6,7\cdot10^5$	$3\cdot10^{10}$	80,17	5,5	$8,4\cdot10^5$	$2\cdot10^{10}$	85,19	0,6	$1.5\cdot10^5$	$1,5\cdot10^{10}$
				80,33	3,8	$2,3\cdot10^5$	$1.5\cdot10^{10}$	72,13	4,7	$1.4\cdot10^5$	$1,5\cdot10^{10}$
				86,40	3,2	$1,5\cdot10^4$	$2.5\cdot10^9$	63.48**	0,5	$4.6\cdot10^4$	$<10^{10}$
				87,2	8,3	$5,5\cdot10^5$	$3,3\cdot10^{10}$	87.28	9,5	$2\cdot10^5$	$2,3\cdot10^{10}$

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*r — distance between shower axis and the cloud chamber (in meters), N — number of particles in the extensive air shower, E_k — kinetic energy of the particle generating the shower in the cloud chamber.

**Event originating in the last plate.

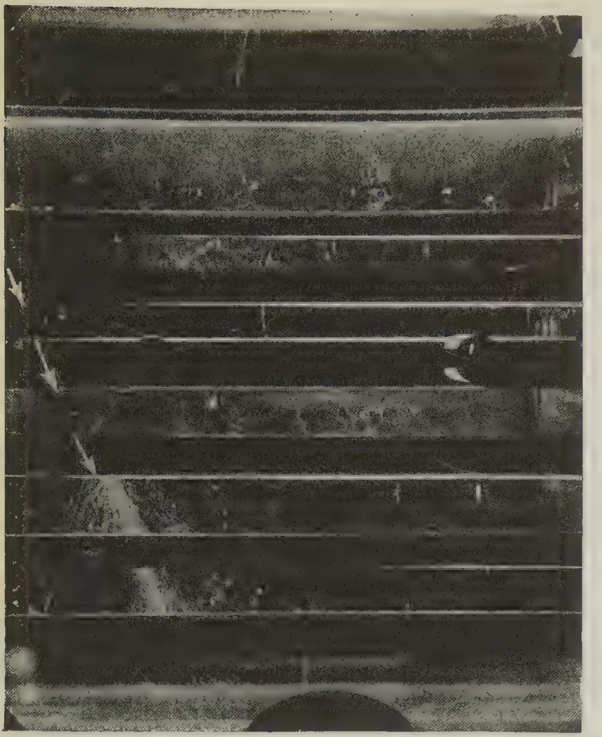

FIG. 2. Electron-nuclear shower generated by a charged particle in the fifth plate of the cloud chamber (case 78,20). Track of the primary particle indicated by arrows.

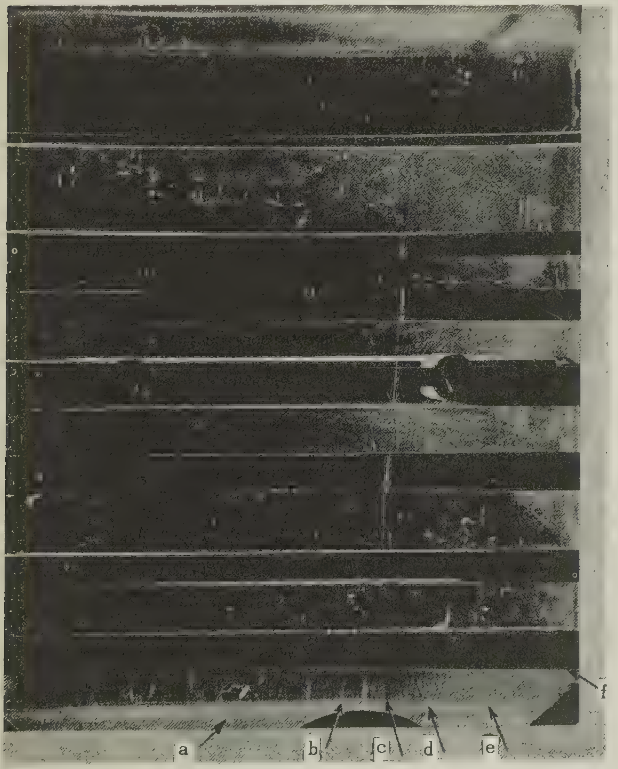

FIG. 3. Electron-nuclear shower consisting of a large number of penetrating particles (a, b, c, d, e, f) (case 22,22). Track of the primary particle indicated by arrows.

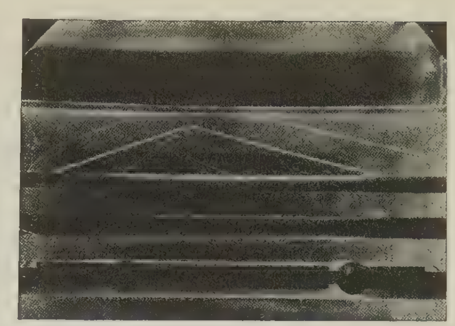

FIG. 4. Electron-nuclear shower produced in the gas by a low-energy particle (case 60,14).

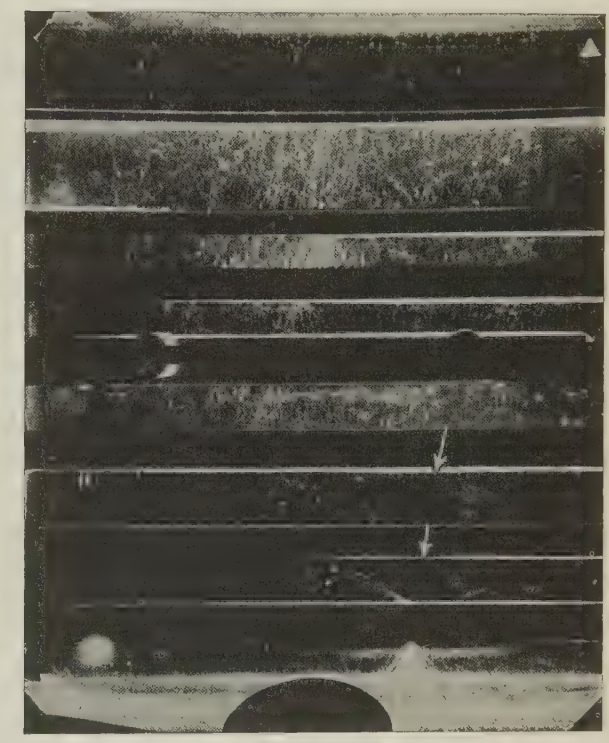

FIG. 5. Electron-nuclear shower generated by the primary particle in the seventh plate of the chamber (case 82,21). Track of the primary particle indicated by arrows.

angle $\theta_{1/2}$ which contains half of the particles in the laboratory system:

$$\sqrt{\gamma_c^2 - 1} = 1/\tan\theta_{1/2} \qquad (1)$$

We used Eq. (1) to calculate γ_c for all showers of the second group and, using the graph in Fig. 1 of Ref. 4, determined the values of kinetic energy E_k of primary particles initiating the showers as well as the possible number n of nucleons involved in the collision. The results are given in Table II. The error in the energy value, due to the error in γ_c, amounts to 30 — 50%.

The values of γ_c calculated according to Eq. (1) might possibly be overvalued because of the

TABLE II. Showers of the second group*

Case no.	N_p	N_{hi}	N_e	$\theta^{\circ}_{1/2}$	γ_c	n	m	E_k (ev)	$\bar{\varepsilon}_{\pi}$, (ev)
22,22	7	0	3—4	48	1,35	5	6	$7 \cdot 10^{9}$	$4{,}9 \cdot 10^{8}$
28,32	12	6	2—4	23	2,6	5	13	$3{,}7 \cdot 10^{10}$	$1{,}9 \cdot 10^{9}$
44,14	9	2	3	42	1,5	6	9	$1{,}2 \cdot 10^{10}$	$6{,}2 \cdot 10^{8}$
63,7	5	2	0	55	1,2	5	4	$4{,}3 \cdot 10^{9}$	$3{,}3 \cdot 10^{8}$
73,39	14	8	1	40	1,55	9	15	$2{,}3 \cdot 10^{10}$	$7{,}6 \cdot 10^{8}$
80,17	7	1	6—8	23	2,6	3	8	$2{,}0 \cdot 10^{10}$	$1{,}4 \cdot 10^{9}$
87,2	11	1	2	27	2,2	5	12	$3{,}3 \cdot 10^{10}$	$1.3 \cdot 10^{9}$
87,28	9	10	6	30	2,0	4	10	$2.3 \cdot 10^{10}$	$1{,}2 \cdot 10^{9}$

*N_p – number of penetrating particles in the shower, N_{hi} – number of heavily ionizing particles in the shower, N_e – number of electrons, n – number of nucleons involved in the collision, m – number of produced π mesons.

assumption that the velocity of all particles in the center-of-mass sytem is close to c. We therefore found γ_c for the showers of the second group in yet another way. The energy of the particles, all assumed to be π mesons, was determined from their range in the lead plates of the cloud chamber. In case of particles not stopping in the chamber, the minimum value of energy was found. A value γ_c for which the distribution in the center-of-mass system became isotropic was then found by means of curves giving, for various γ_c, the relation between the angle in the laboratory system and the angle in the center-of-mass system for particles with various momentum. Such a value represented the lower limit of γ_c since, for the case of many particles a minimum energy was assumed instead of the real value. The values for γ_c obtained using the above method were, within limits of accuracy, identical with those obtained from Eq. (1).

The absence of high-energy electron cascades in showers of the second group was explained calculating the mean energy of π mesons under the assumption that the kinetic energy is divided evenly between the particles at the moment of disintegration. The values of the mean energy of π mesons are given in the last column of Table II.

It can be seen from the table that the mean energy of π mesons is relatively small in the cases 22,22; 44,14; and 63,7. It should be remembered also that the energy of photons originating in π^0 decay is, in the mean, smaller by a factor of 2. All this could explain the absence of dense electronic cascades when a few electrons only are present directly under the plate in which the interaction occurred. The approximate number of such electrons is given in column 4 of the table. In the cases 80,17 and 87,28 the mean energy of the π mesons is quite large. The interactions, however, occurred near the border of the illuminated region of the chamber and the continuation of the electronic cascades could have well developed beyond the visible region; 6 — 8 electrons are observed below the plate in which the interaction took place. Finally, in the cases 28,32; 73,39; and 87,2 the absence of large electronic cascades can be explained either by fluctuations in the ratio of charged and neutral π mesons or by fluctuations in energy distribution, favoring the charged mesons. It can be seen from the table that in those cases the largest number of charged penetrating particles N_p is observed. If the primary energy is recalculated assuming that only one neutral π meson was produced, the energy of the nuclear-active particle decreases only by 20 — 30%.

It was impossible to determine the energy of the showers of the first group in the same way since a part of penetrating particles might have been hidden in dense electronic cascades. Consequently, it was impossible to determine the velocity of the center-of-mass system. The energy of the primary particles generating the showers of the first group was, therefore, found from the energy of the electronic cascades. As it was mentioned above, the relatively small number of penetrating particles in these showers and their narrow angular distribution made it possible to assume that the showers originated in nucleon-nucleon collisions. If we assume that the relation between the mean energies of the nucleons and π mesons obtained for the showers of the second group remains valid for the first group, i.e., that in the mean a nucleon carries away an amount of energy three times larger than a π meson, then the energy carried away by all π^0 mesons can be written as follows:

$$E_{\pi^0} = \frac{m}{3}\,\frac{E_L}{3n+m} = \frac{m}{3}\,\frac{\gamma_c E_0}{3n+m},$$

where E_0 is the total energy in the center-of-mass system, E_L is the total energy in the laboratory system ($E_L = E_k + nM_0$, where M_0 is the nucleonic mass), m is the number of π mesons, n is the number of nucleons (in the above case

TABLE III. Showers of the first group

Case no.	E_{π^0}, (ev)	N_p	N_{hi}	$\theta^*_{\max}$,*	$n=2$			$n=3$		
					E_K, (ev)	m	γ_c	E_K, (ev)	m	γ_c
3,30	$2{,}5 \cdot 10^9$				$1.5 \cdot 10^{10}$	5	3,0			
18,29	$3 \cdot 10^9$	2—3	2	62	$1.5 \cdot 10^{10}$	5	3,0	$1{,}6 \cdot 10^{10}$	7	2,2
36,12	$5 \cdot 10^9$		—	—	$3 \cdot 10^{10}$	7	4,0			
40,16	10^9	1	—	25	$6 \cdot 10^9$	3	2,0			
45,10	$5 \cdot 10^9$				$3 \cdot 10^{10}$	7	4,0			
56,1	10^9		2		$6 \cdot 10^9$	3	2,0			
70,37	$2{,}5 \cdot 10^9$	$\geqslant 5$	1—2	47	$1{,}5 \cdot 10^{10}$	5	3,0	$1{,}6 \cdot 10^{10}$	7	2,2
72,13	$3{,}10^9$	$\geqslant 3$	1	48	$1{,}5 \cdot 10^{10}$	5	3,0	$1{,}6 \cdot 10^{10}$	7	2,2
76,51	$3 \cdot 10^9$	>1	1	30	$1{,}5 \cdot 10^{10}$	5	3,0			
78,20	$5 \cdot 10^9$	$\geqslant 4$	1	40	$3 \cdot 10^{10}$	7	4,0	$2{,}8 \cdot 10^{10}$	9	2,8
80,33	$3 \cdot 10^3$	$\geqslant 2$			$1{,}5 \cdot 10^{10}$	5	3,0			
85,19	$3 \cdot 10^9$			38	$1{,}5 \cdot 10^{10}$	5	3,0			

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*$\theta_{\max}$ – maximum angle of emission of a penetrating particle with respect to direction of the primary in the laboratory system.

n = 2).

Using the above formula and the curve for nucleon-nucleon collisions from Fig. 1 of Ref. 4, it is possible to find the energy of the shower generating primary from the energy of the electronic cascade. The energy of electronic cascades was determined from cascade curves[6] using the method described in Ref. 7. Energies of primary particles generating the showers of the first group are given, together with other data on the showers, in Table III.

It should be noted that among the showers of the first group some may be due to collisions with two or more nucleons of the target nucleus. Such may be, e.g., the cases 18,29; 70,37; 72,13; and 78,20 in which a relatively large number of penetrating particles is observed. It should be noted that only penetrating particles emitted at large angles to the direction of the primary are given in Table III, since particles going in the core are hidden by dense electron cascades. The energy of the primary particle can be found by a similar method assuming that three nucleons were involved in the collision. The values of γ_c and E'_k obtained in that way for the four events are given in Table III. It can be seen that the assumption that the collision involved two nucleons of the nucleus, instead of one only, hardly changes the value for the primary energy. Only the number of π mesons produced and the value of γ_c are affected.

Corresponding to the narrow angular distribution of particles in showers of the first group the values of γ_c are larger than is the case for showers of the second group.

Showers of the third group were generated, clearly, by particles with energy close to the electron-nuclear shower production threshold. The energy of the primaries was therefore determined approximately from the number of charged particles in the shower using the curves of Camerini et al.[8] which give the dependence of the number of relativistic, grey, and black tracks on the energy of the primary particle. The results are given in Table IV.

TABLE IV.

Showers of the third group

Case no.	N_p	N_{hi}	E_k
57,10	—	2	$\sim 10^9$
60,14	1	3	$\sim 2 \cdot 10^9$
75,4	1	4	$\sim 10^9$
86,40	2	—	$2{,}5 \cdot 10^9$

Determination of the energy of primaries was especially difficult for showers of the fourth group since it was impossible to discern between penetrating particles and electrons. The energy was estimated as follows: it was assumed that all relativistic particles were electrons and that the interaction occurred in the middle of the plate. The energy carried away by π^0 mesons was found from cascade curves. The primary particle energy was then determined by a method similar to that used for the showers of the first group. Such an estimate is very crude since, firstly, penetrating particles may be present among relativistic particles and, secondly, the energy of electron cascade is not given very accurately by the number of electrons at a single — and at that uncertain — depth. The results, nevertheless, agree in the order of magnitude with the values obtained for the other shower groups and there are therefore reasons to believe that they do not differ more from the true values than by a factor of 2 or 3.

The energies of nuclear-active particles generating showers in the lead plates of the cloud chamber are given in Table I.

The integral energy spectrum of nuclear-active particles is given in Fig. 6. The spectrum refers to extensive air showers with number of particles in the range $10^4 - 10^6$ ($\bar{N} = 1.4 \times 10^5$) and distances < 10 m from shower axis ($\bar{r} = 3.7$ m).

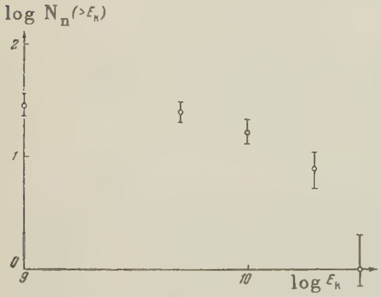

FIG. 6. Integral spectrum of nuclear-active particles in extensive air showers. $N_n(> E_k)$ is the number of nuclear-active particles with energy > E_k.

4. NATURE OF NUCLEAR-ACTIVE PARTICLES

In 19 cases out of the 30 recorded nuclear interactions it was possible to determine if the generating particles were charged or neutral. In 11 cases no conclusions could be drawn concerning the nature of primary because of unfavorable position of the shower in the cloud chamber (close to the glass or to the walls, in a dense electron cluster). The charge of the primary, for the cases when it could be ascertained, is given in Table I.

It can be seen that about half of the nuclear-active particles are charged and half neutral. This leads naturally to the assumption that at sea level the nuclear-active component of extensive air showers with energies of $10^9 - 10^{11}$ ev consists mainly of nucleons.

In conclusion the authors wish to express their gratitude to G. T. Zatsepin and D. S. Chernavskii for discussion of results and valuable advice. A. T. Abrosimov, V. I. Goman' kov, G. P. Kulikov, I. V. Rakovol' skaia, D. F. Rakitin, and V. L. Rozin took part in various stages of the work.

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SOVIET PHYSICS JETP VOLUME 34(7), NUMBER 1 JULY, 1958

ELASTIC SCATTERING OF 580-MEV NEUTRONS BY PROTONS AT SMALL ANGLES

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The differential cross section for scattering of 580 Mev neutrons by protons was measured at angles of 11.25 to 35° (in the c.m.s.). The cross section was observed to rise rapidly with decrease of the scattering angle. The results indicate that the probabilities for scattering in the forward and backward directions are approximately equal.

THE present work is a continuation of investigations of the scattering of neutrons having an average energy of 580 Mev against protons,[1] with particular interest in elastic n-p scattering at neutron energies considerably above the threshold energy for meson production. The experiment consists in measuring the differential n-p scattering cross section in the small angle region ($\vartheta \leq 35°$ in c.m.s.). The usual method for measuring the differential cross section for elastic n-p scattering by counting recoil protons in a given angular region is not convenient because of the small energy of the recoil protons. Because of this fact, the measurements described here were obtained by a method which differs from that of Ref. 1, the basic conditions of the experiment, however (energy threshold, angular resolution, and detector solid angle), are still the same as in Ref. 1.

EXPERIMENTAL ARRANGEMENT

The differential n-p scattering cross section was obtained by measuring the number of neutrons scattered at a given angle from a hydrogen target placed in the path of the neutron beam. The high energy neutrons are obtained from charge exchange scattering of 680 Mev protons against a beryllium target placed in the chamber of the synchrocyclotron of the Laboratory for Nuclear Problems (Joint Institute for Nuclear Research). The energy distribution curve for the neutrons has a maximum at 600 Mev and a half-width of 130 Mev (Rev. 2). Taking into account the threshold of the neutron detector (450 Mev) the mean effective energy of the neutrons turns out to be 580 Mev. The intensity of the neutron beam at the position of the scatterer is 2×10^4 neutrons/sec-cm^2.

The first experiments were carried out with polyethylene and graphite scatterers. The main experimental data was then obtained using a scatterer of liquid hydrogen contained in a cylindrical glass Dewar flask. The Dewar flask has a capacity of three liters and measures 10 cm in diameter and 34 cm in height; it is encased in a protective duraluminum jacket; the beam passes through a window consisting of an aluminum foil 20 μ thick. The structure of the target allows an experiment with liquid hydrogen to proceed during seven hours. The effect of elastic n-p scattering was determined in the first case by comparing results for polyethylene and carbon, and in the second case by comparing results obtained with an empty Dewar flask and one filled with liquid hydrogen; the small effect (< 3%) due to air filling the Dewar flask after evaporation of the liquid hydrogen was neglected.

The neutron detector used in this experiment for measuring the flux of scattered neutrons is made of five scintillation counters and a "converter" consisting of a polyethylene cylinder having a thickness of 5.6 gm/cm^2 and a diameter of 6 cm, placed between the first two counters (Fig. 1). The neutrons falling on the converter undergo exchange scattering. The recoil proton leaving the converter are detected by the last four counters. The first counter is used to exclude charged particles arriving from the target and operates in anticoincidence with the remaining counters. In order to decrease overloading in the anticoincidence scheme, counter 1 is originally connected in coincidence with counter 2, and then pulses resulting from their coincidences are used in the anticoincidence scheme (Fig. 1). The scintillators consist of a solution of terphyl $(C_6H_5)_2C_6H_4$ in phenycyclohexane $C_6H_5CH(CH_2)_5$ with a concentration of 3 gm/ℓ, filling plexiglass containers which meas-

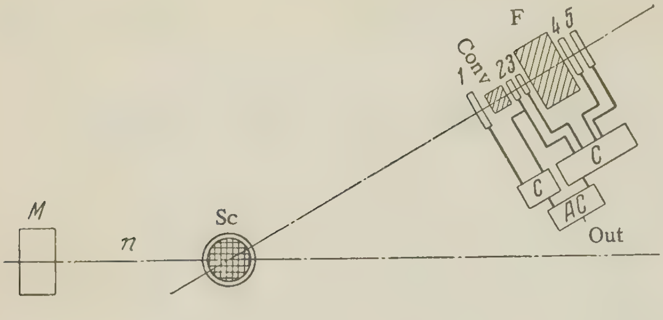

FIG. 1. Experimental arrangement. n – neutron beam; M – monitor; Sc – scatterer (Dewar flask with liquid hydrogen); 1, 2, 3, 4, 5: scintillation counters; Conv – converter; F – filter; C – coincidence arrangement; AC – anticoincidence arrangement; Out – output.

ure as follows: (1) 12 × 12 × 1 cm, (2) 5 × 5 × 1 cm, (3) 5 × 5 × 1 cm, (4) 12 × 12 × 1 cm, and (5) 12 × 12 × 1 cm. The energy threshold of the neutron detector is set by a copper filter placed between counters 3 and 4. In computing the thickness of the filter for scattering angles $\Phi = 15, 10, 5°$ (in the laboratory system), it was assumed on the basis of the results obtained in Refs. 2 and 3, that the average energy loss due to charge exchange of the neutron in light materials is about 15% of the initial energy and remains constant in a large energy interval (170 – 680 Mev). Estimates indicate that the average effective energy of neutrons scattered at small angles of 15, 10, 5° (in the laboratory system) changes by a relatively small amount (in our case 40 Mev), which permits one to assume that the neutron detector has constant efficiency over the given angular interval. The angular resolution of the detector is 2°.

In measuring the differential cross section for elastic n-p scattering, the scatterer is placed in the path of the neutron beam, the neutron detector is set up at a given angle with respect to the direction of the beam and one measures for a given scatterer the difference of the counting rate with the converter placed between the first and second counters, and with no converter. The relative differential cross section is then found from the formula:

$$(H_{\mathrm{H}_2}^{\mathrm{C}} - N_{\mathrm{H}_2}^{0}) - (N_{\mathrm{d}}^{\mathrm{C}} - N_{\mathrm{d}}^{0}) = n\sigma_{np}(\Phi), \tag{1}$$

where $N_{\mathrm{H}_2}^{\mathrm{C}}$, $N_{\mathrm{H}_2}^{0}$, $N_{\mathrm{d}}^{\mathrm{C}}$, N_{d}^{0} are respectively the counting rates with and without converter, when the Dewar flask is filled with hydrogen, and when it is empty; $\sigma_{\mathrm{np}}(\Phi)$ is the differential cross section at a given angle Φ (in the laboratory system); n is a proportionality factor which depends on the geometry of the detector and the intensity of the neutron beam (it remains constant under the experimental conditions).

Thus the measurement of the neutron count scattered at a given angle allows one to compute the differential cross section for n-p scattering in relative units at angles of 5, 10, 15° (in the laboratory system). The absolute value of the cross section was obtained from the knowledge of the cross section at 15° found in Ref. 1. The counting rate $N_{\mathrm{H}_2}^{0}$ was 50% of the counting rate $N_{\mathrm{H}_2}^{\mathrm{C}}$ at angles of 15° and 10°, and 60% at an angle of 5°. Background due to accidental coincidences did not exceed 3%.

It must be noted that the small efficiency of the neutron detector (1 – 2%) imposes severe conditions on the efficiency of the anticoincidence scheme. Indeed, a change in position of the converter, when measuring with and without it, noticeably (about 20 Mev in our case) changes the detector threshold for charged particles which are erroneously allowed by the anticoincidence channel and because of this, registered by the detector. An estimate based on a known neutron spectrum at the given converter thickness (5.6 g/cm^2 CH$_2$), indicates that the relative increase in the number of charged particle due to the indicated change in detector threshold, is $\Delta N/N = 1/19(1 - \epsilon)$, where ϵ is the efficiency of the anticoincidence channel.

The efficiency of the anticoincidence channel was obtained in a special experiment on the 660 Mev proton beam and turned out to be 99.9 ± 0.1%. Therefore in our case, $\Delta N/N = 0.005\%$, and the error in the measured number of registered neutrons due to inefficiency of the anticoincidence channel did not exceed one percent and was thus neglected.

RESULTS OF THE MEASUREMENTS

The differential scattering cross section measured at angles of 11° and 23° (in the c.m.s.) were found to be as follows: $\sigma_{\mathrm{np}}(11°) = (7.5 \pm 1) \times 10^{-27}$ cm^2/sterad and $\sigma_{\mathrm{np}}(23°) = (5 \pm 0.8) \times 10^{-27}$ cm^2/sterad. The indicated errors are due to the standard statistical deviations. As in Ref. 1, the uncertainty in the total cross section for elastic n-p scattering from which the differential scattering cross section is normalized, is not included in the indicated error.

The differential cross section for elastic scattering of 580 Mev neutrons against protons, $\sigma_{\mathrm{np}}(\vartheta)$ in the angular region 11° to 180° (in the c.m.s.) is shown in Fig. 2. For the sake of comparison, the results of n-p scattering at 90 Mev (Ref. 4) and 400 Mev (Ref. 5) are also shown.

The behavior of $\sigma_{np}(\vartheta)$ in the region of small angles ($\vartheta \leq 35°$) at 580 Mev, shows a strong increase in differential cross section with decrease in the scattering angle. Comparison of the data shown in Fig. 2 indicates that the character of the scattering in this angular region varies considerably as the neutron energy grows from 400 to 580 Mev. The approximate constancy of the scattering cross section for angles $\vartheta \leq 35°$ observed at 400 Mev, disappears and the symmetry in "forward" ($\vartheta \sim 0°$) and "backward" ($\vartheta \sim 180°$) cross section which is present at an energy of 90 Mev, is reestablished to a considerable extent. The ratio of the scattering cross section at angles of 10° and 170°, $\sigma_{np}(170°)/\sigma_{np}(10°)$, equals 1, 2.4 and 8 at energies of 90, 400, and 580 Mev. Thus the relation between "ordinary" and "exchange" scattering at 580 Mev, apparently approaches the relation which exists at 90 Mev, when the curve for $\sigma_{np}(\vartheta)$ is symmetrical about $\vartheta = 90°$ (in the c.m.s.), and the two types of scattering contribute about equally to the total interaction cross section. This change in the character of the elastic scattering of neutrons against protons in the angular region $\vartheta \leq 35°$, is apparently related to the growth of the probability for the process of π meson production in n-p collision, with increase in nucleon energy from 400 to 580 Mev (Ref. 6).

It is interesting that the increase in the differential cross section at small angles near 0° as the energy grows from 400 to 580 Mev is predicted by the optical theorem[7]

$$\sigma_{np}(0) \geqslant k^2\sigma_t^2/16\pi^2, \qquad (2)$$

where $\sigma_{np}(0)$ is the differential cross section for elastic n-p scattering at 0° (c.m.s.); k is the wave number of the incident neutron; σ_t is the total n-p interaction cross section. The smallest possible value of $\sigma_{np}(0)$ at 580 Mev is about $6 \times 10^{-27}\,\mathrm{cm}^2$ according to the inequality (2), while the experimental value of $\sigma_{np}(0)$ at 400 Mev does not exceed $4 \times 10^{-27}\,\mathrm{cm}^2$ (Fig. 2). Therefore the differential cross section for elastic scattering of neutrons against protons at 0° (c.m.s.) must go through a minimum in the energy range 300 — 400 Mev, and must increase as the energy of the colliding nucleons goes up.

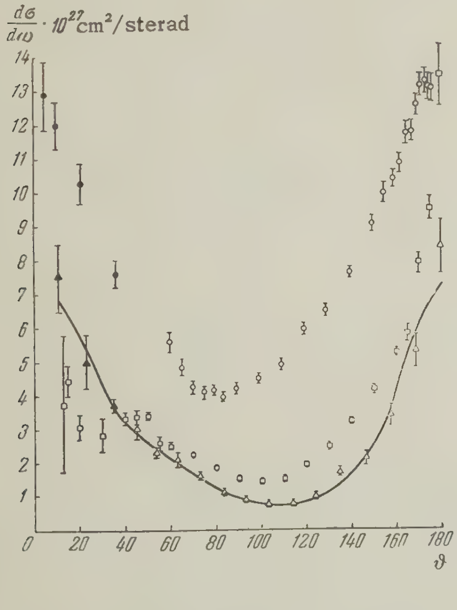

FIG. 2. Differential cross section for elastic n — p scattering. ●, ○ — Ref. 5, E_n = 91 Mev; □ — Ref. 4, E_n = 400 Mev; Δ — Ref. 1, E_n = 580 Mev; ▲ — Present work. Solid curve — graph of the approximate expression (3).

In Ref. 1, an approximation was obtained by least squares to $\sigma_{np}(\vartheta)$ in the form of a series of Legendre polynomials up to and including L = 12. The behavior of $\sigma_{np}(\vartheta)$ for $\vartheta \leq 35°$, found in the present work, noticeably changes the character of the approximate expression which in our case has the form

$$\begin{aligned}\sigma_{np}(\vartheta) = ƛ^2 [1.38 \pm 0{,}07 &- (0.7 \pm 0.08)\, P_1(\cos\vartheta) \\ + (1{,}92 \pm 0.08)\, P_2(\cos\vartheta) &- (0{,}12 \pm 0{,}09)\, P_3(\cos\vartheta) \\ + (0{,}92 \pm 0{,}1)\, P_4(\cos\vartheta) &- (0.03 \pm 0.1)\, P_5(\cos\vartheta) \\ + (0.58 \pm 0{,}11)\, P_6(\cos\vartheta) &- (0{,}15 \pm 0{,}12)\, P_7(\cos\vartheta) \\ + (0.33 \pm 0.14)\, P_8(\cos\vartheta) &- (0{,}23 \pm 0{,}15)\, P_9(\cos\vartheta) \\ + (0.11 \pm 0{,}15)\, P_{10}(\cos\vartheta) &- (0{,}04 \pm 0{,}15)\, P_{11}(\cos\vartheta) \\ &- (0.14 \pm 0{,}11)\, P_{12}(\cos\vartheta)], \end{aligned} \qquad (3)$$

where ƛ is the wave length of the nucleon in the center-of-mass system.

The discovery that the probability for scattering is about the same at $\vartheta = 0°$ and 180° leads to the fact that the odd order terms in the polynomial expansion have small coefficient. Deviations from symmetry with respect to $\vartheta = 90°$ are characterized in the given expression by the main term containing $P_1(\cos\vartheta)$. Contributions to the scattering cross section of interference waves corresponding to isotopic spin states T = 0 and T = 1 of the n-p system are much smaller for $\vartheta = 0°$ and 180° in the 580 Mev region than in the regions of 300 — 400 Mev where a strong asymmetry is observed between forward ($\vartheta = 0°$) and backward ($\vartheta = 90°$) scattering.

It is worth noting that an estimate of the accuracy of the expansion (3) is given by the criterion[8]

$$\sum_i \Big[\sigma_{np}(\vartheta_i) - \sum_L \alpha_L P_L(\cos\vartheta)\Big]^2 \Big/ (\Delta\sigma_i)^2 = m - l; \qquad (4)$$

$\sigma_{np}(\vartheta_i)$ is here the measured value of the differential cross section at the point ϑ_i; $\sum\limits_L \alpha_L P_L(\cos\vartheta)$ is the cross section computed from expansion (3); $(\Delta\sigma_i)^2$ is the weight of a given measurement, $\Delta\sigma_i$ being the experimental error; m is the number of points at which the cross section is measured, and ℓ the number of coefficients in (3); such an estimate does not yield a completely satisfactory result. It is found that the sum of the squares of the deviations of the computed curve from the measured values of the cross section which appear on the left-hand side of Eq. (4), equal 7.7 when $\ell = 4$. The largest deviations are found in the angular region around 180°, which apparently indicates the need for increasing the number of terms in the expansion (3). However, the magnitude of the errors in the coefficients α_L of expansion (3), shows that in view of the existing experimental uncertainty, an increase in the number of terms for expansion (3) is hardly sensible.

Finally it must be noted that comparison of the present results with the data on elastic p-p scattering at the same energy[9] confirms the hypothesis of isotopic invariance.

The authors consider it their pleasant duty to thank Prof. V. P. Dzhelepov for discussing the results of this work, S. A. Baldin for carrying out preliminary measurements whose results have permitted choosing optimal experimental conditions, Iu. N. Simonov and G. N. Tentiukov for helping with the experiment.

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*A NEW METHOD IN THE THEORY OF SUPERCONDUCTIVITY. I**

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The canonical transformation method previously developed by the author for the theory of superfluids is generalized in the present paper. By application of this method and the principle of compensation of "dangerous" diagrams, it is shown that a superconducting state is inherent in the Fröhlich model. Computation of the principal parameters of this state leads to formulas that confirm those of the theory of Bardeen, Cooper, and Shrieffer.

AFTER the discovery of the isotopic effect, the idea became universal that the interaction between the electrons and the lattice must play a fundamental role in the phenomenon of superconductivity. A series of very interesting syntheses,[1-4] which treat a system of electrons interacting with the phonon field have already been carried out. In the present paper, we shall show that, by further development of a method which we have put forward previously for the theory of superfluidity, it is possible to give a systematic theory of the superconducting state. In this case, in particular, the results of the theory of Bardeen, Cooper, and Schrieffer[3] are confirmed.

For simplicity, we shall start out from the model proposed by Fröhlich,[1] in which the Coulomb interaction is not introduced explicitly and the dynamic system is characterized by the Hamiltonian†

$$H_{\mathrm{Fr}} = \sum_{k,\, s} E(k)\, a_{ks}^{+} a_{ks} + \sum_{q} \omega(q)\, b_q^{+} b_q + H',$$

$$H' = \sum_{\substack{k,\, q,\, s \\ k'-k=q}} g \left\{ \frac{\omega(q)}{2V} \right\}^{1/2} a_{ks}^{+} a_{k's} b_q^{+} + \text{conjugate}. \qquad (1)$$

where E(k) is the energy of the electron, $\omega(q)$ is the energy of the phonon, g is the coupling constant, and V the volume.

As is now well known, ordinary perturbation theory in terms of powers of the coupling constant is inapplicable, since the electron-phonon interaction, in spite of its smallness, is seen to be very considerable when close to the Fermi surface. Therefore, we shall first carry out a certain canonical transformation, starting out from the following considerations.

First, we note that the matrix elements, corresponding to virtual creation of "particles" from the vacuum, are always associated with the energy denominators

$$\{\varepsilon(k_1) + \ldots + \varepsilon(k_{2s}) + \omega(q_1) + \ldots + \omega(q_r)\}^{-1},$$

in which $\epsilon(k) \sim |E(k) - E_F|$ is the energy of the particle [electron — $E(k) > E_F$ or hole — $E(k) < E_F$] which becomes small at the Fermi surface.

Such denominators are not generally "dangerous," and do not lead to singularities in integration over the momenta $k_1, \ldots, k_{2s}$, $q_1, \ldots, q_r$, except when we deal with a virtual process of the creation of a single pair, without phonons. Then, because of the conservation law, the particles of this pair will have oppositely directed momenta $\pm k$, and the energy denominator $\frac{1}{2}\epsilon(k)$ will become "dangerous" for integration. We note further that their spins will also be opposite.

Thus, in the choice of the canonical transformation, it must be kept in mind that it is necessary to guarantee the mutual compensation of the diagrams which lead to virtual creation from the vacuum of pairs of particles with opposite momenta and spins.

We now underscore the analogy with the situation holding in our theory of superfluids, namely, in the consideration of a non-ideal Bose gas, where virtual creation of a pair of particles (with momenta ± k) from the condensate plays the same role. We then made use of a linear transformation of the Bose amplitudes, "mixing" b_q with b^+_{-q}.

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*Note added in proof (Dec. 20, 1957). Very recently, when our papers had already gone to press, the manuscript of the detailed work of Bardeen, Cooper, and Schrieffer became known to us. In particular, a derivation is given therein for formulas which we have cited from the brief letter by the same authors. We must note that there are several places that are not clear in the formulation of the method of Bardeen, Cooper, and Schrieffer. A detailed comparison of the two methods will be given in a paper which is currently being prepared for press.

†A system of units is employed here in which ħ = 1.

Generalizing this transformation, we now bring into consideration the case of new Fermi amplitudes

$$\alpha_{k0} = u_k a_{k,\,{}^1\!/_2} - v_k a^+_{-k,\,-{}^1\!/_2},$$
$$\alpha_{k1} = u_k a_{-k,\,-{}^1\!/_2} + v_k a^+_{k,\,{}^1\!/_2}$$

or

$$a_{k,\,{}^1\!/_2} = u_k \alpha_{k0} + v_k \alpha^+_{k1},$$
$$a_{-k,\,-{}^1\!/_2} = u_k \alpha_{k1} - v_k \alpha^+_{k0},$$

where u_k, v_k are real numbers, connected by the relation

$$u_k^2 + v_k^2 = 1.$$

It is not difficult to check that such a transformation preserves all the commutation properties of the Fermi operators and is therefore canonical.

We further note that it represents a generalization of the ordinary transformation, with the help of which the creation and annihilation operators of holes inside the Fermi surface and of electrons outside this surface are introduced. Actually, if we set

$$u_k = 1,\ v_k = 0 \ \text{ for } \ E(k) > E_F$$
$$u_k = 0,\ v_k = 1 \ \text{ for } \ E(k) < E_F,$$

then we obtain

$$\alpha_{k0} = a_{k,\,{}^1\!/_2},\ \alpha_{k1} = a_{-k,\,-{}^1\!/_2} \ \text{ for } \ E(k) > E_F,$$
$$\alpha_{k0} = -a^+_{-k,\,-{}^1\!/_2},\ \alpha_{k1} = a^+_{k,\,{}^1\!/_2} \ \text{ for } \ E(k) < E_F,$$

so that, for example, α_{k0} will be an annihilation operator for an electron of momentum k and spin $\frac{1}{2}$ outisde the Fermi sphere, and will be an annihilation operator for a hole with momentum −k and spin $-\frac{1}{2}$ inside.

In the general case, when $(u_k, v_k) \neq (0, 1)$, we have to deal with the superposition of a hole and an electron.

Generalizing to the consideration of the Fröhlich Hamiltonian, we note that it will be more advantageous to us not to restrict ourselves to the relation

$$\sum_{k,\,s} a^+_{ks} a_{ks} = N_0,$$

where N_0 is the total number of electrons; we therefore make use of the ordinary procedure in such a situation and introduce a parameter λ which plays the role of the chemical potential. Then, instead of H_{Fr}, we have the Hamiltonian

$$H = H_{Fr} - \lambda N. \qquad (2)$$

We determine the parameter λ in turn from the condition that in the state under examination

$$\overline{N} = N_0. \qquad (3)$$

Transforming H to the new Fermi amplitudes, we get

$$H = U + H_0 + H_{\text{int}},\ H_{\text{int}} = H_1 + H_2 + H_3,$$

where U is the constant

$$U = 2\sum E(k)\, v_k^2 - 2\lambda \sum v_k^2,$$

and where

$$H_1 = \sum_{\substack{k,\,k' \\ k'-k=q}} g\sqrt{\frac{\omega(q)}{2V}}\,\{u_k v_{k'} \alpha^+_{k0} \alpha^+_{k'1} + u_{-k} v_{-k'} \alpha^+_{-k'0}\, \alpha^+_{-k1}$$
$$+ u_{k'} v_k \alpha_{k1} \alpha_{k'0} + u_{-k'} v_{-k} \alpha_{-k'1} \alpha_{-k0}\}\, b^+_q + \text{conjugate}$$

$$H_2 = \sum_{\substack{k,\,k' \\ k'-k=q}} g\sqrt{\frac{\omega(q)}{2V}}\,\{u_k u_{k'} \alpha^+_{k0} \alpha_{k'0} + u_{-k} u_{-k'}\, \alpha^+_{-k1'}\, \alpha_{-k'1}$$
$$- v_k v_{k'} \alpha^+_{k'1} \alpha_{k1} - v_{-k} v_{-k'} \alpha^+_{-k'0}\, \alpha_{-k0}\}\, b^+_q + \text{conjugate}$$

$$H_3 = \sum 2\,(E(k) - \lambda)\, u_k v_k\, (\alpha^+_{k0}\, \alpha^+_{k1} + \alpha_{k1}\, \alpha_{k0}),$$

$$H_0 = \sum (E(k) - \lambda)\,(u_k^2 - v_k^2)\,(\alpha^+_{k0}\, \alpha_{k0} + \alpha^+_{k1}\, \alpha_{k1})$$
$$+ \sum \omega(q)\, b^+_q\, b_q.$$

We introduce the filling number

$$\nu_{k0} = \alpha^+_{k0}\, \alpha_{k0},\ \nu_{k1} = \alpha^+_{k1}\, \alpha_{k1}$$

of new quasi-particles, which are generated by the operator α^+. Then the "interaction-free vacuum," i.e., the state C_V in which

$$H_0 C_V = 0,$$

will evidently be

$$C_V = \prod_k \delta(\nu_{k0})\, \delta(\nu_{k1})$$

with zero values of the numbers ν.

We further note that λ ought to be close to E_F, since $\lambda = E_F$ in the absence of interaction, and that consequently, the expression

$$\varepsilon(k) = (E(k) - \lambda)\,(u_k^2 - v_k^2)$$

ought to vanish on a surface close to the Fermi surface.

We now see that the virtual process of creation from the vacuum of a pair of quasi-particles ν_{k0} and ν_{k1} without phonons will be "dangerous" in the sense of the criterion given earlier, since the corresponding energy denominator will be

$$1/2\varepsilon(k).$$

The Hamiltonian H_3, which, being applied to a vacuum, gives the diagram of Fig. 1,* leads directly to such a process. This same process is also obtained, moreover, as a result of the combined action of H_1, H_2. Thus, for example, in the second order in the coupling constant g, we have the diagrams shown in Fig. 2a.

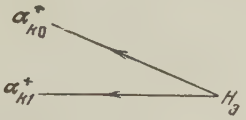

FIG. 1

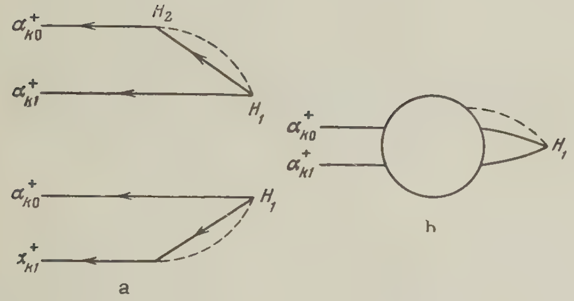

FIG. 2

In higher orders, we get diagrams of the type of 2b, where the circle denotes the coupled part, which cannot be divided into two coupled parts, and which is bound only by the two lines of one pair under consideration.

Making use of the principle of compensation of dangerous diagrams introduced above, we must equate to zero the sum of contributions from the diagrams of Fig. 1 and Fig. 2. We thus obtain an equation for u_k and v_k.

Now we no longer need to take into consideration the diagrams of Figs. 1 and 2 (and their conjugates), and therefore, in the expansions of perturbation theory, no expression will appear with a dangerous energy denominator.

We now construct a second-order equation for u_k and v_k. In this approximation, we must compensate the diagram of Fig. 1 by the diagrams of Fig. 2a. We get

$$2\,(E\,(k)-\lambda)\,u_k v_k+\Omega_k=0,$$

where Ω_k is the coefficient for $\overset{+}{\alpha}_{k0}\overset{+}{\alpha}_{k1}C_V$ in the expression

$$-H_2 H_0^{-1} H_1 C_V.$$

Expanding it, we finally obtain

$$\{\widetilde{E}\,(k)-\lambda\}\,u_k v_k$$

$$=(u_k^2-v_k^2)\,\frac{1}{2V}\sum_{k'} g^2\,\frac{\omega\,(k-k')}{\omega\,(k-k')+\varepsilon\,(k)+\varepsilon\,(k')}\,u_{k'}v_{k'}, \quad (4)$$

where

$$\widetilde{E}\,(k)=E\,(k)-\frac{1}{2V}\sum_{k'} g^2\,\frac{\omega\,(k-k')}{\omega\,(k-k')+\varepsilon\,(k)+\varepsilon\,(k')}\,(u_{k'}^2-v_{k'}^2). \quad (5)$$

Staying within the limits of the approximation, we replace

$$\varepsilon\,(k)=\{E\,(k)-\lambda\}\,(u_k^2-v_k^2)$$

in the denominator of the right side by

$$\widetilde{\varepsilon}\,(k)=\{\widetilde{E}\,(k)-\lambda\}\,(u_k^2-v_k^2).$$

Then, setting

$$\widetilde{E}\,(k)-\lambda=\xi\,(k),$$

we write down the equation in the form

$$\xi\,(k)\,u_k v_k=(u_k^2-v_k^2)$$

$$\times\frac{1}{2\,(2\pi)^3}\int g^2\,\frac{\omega\,(k-k')}{\omega\,(k-k')+\widetilde{\varepsilon}\,(k)+\widetilde{\varepsilon}\,(k')}\,u_{k'}v_{k'}d\mathbf{k}'. \quad (6)$$

This equation obviously contains the trivial solution

$$uv=0, \quad (u,\,v)=(0,1),$$

which corresponds to the "normal state." It possesses, however, an additional solution of another type which goes over into the trivial one at great distances from the Fermi surface.

Denoting

$$C\,(k)=\frac{1}{(2\pi)^3}\int g^2\,\frac{\omega\,(k-k')}{\omega\,(k-k')+\widetilde{\varepsilon}\,(k)+\widetilde{\varepsilon}\,(k')}\,u_{k'}v_{k'}d\mathbf{k}',$$

we find from (6)

$$u_k^2=\frac{1}{2}\left\{1+\frac{\xi\,(k)}{\sqrt{C^2\,(k)+\xi^2\,(k)}}\right\};$$

$$v_k^2=\frac{1}{2}\left\{1-\frac{\xi\,(k)}{\sqrt{C^2\,(k)+\xi^2\,(k)}}\right\}, \quad (7)$$

whence

$$u_k v_k=\frac{C\,(k)}{2\sqrt{C^2\,(k)+\xi^2\,(k)}}; \quad \widetilde{\varepsilon}=\frac{\xi^2\,(k)}{\sqrt{C^2\,(k)+\xi^2\,(k)}}.$$

Thus, our equation reduces to the following form:

$$C\,(k)=\frac{1}{2\,(2\pi)^3}\int g^2\,\frac{\omega\,(k-k')}{\omega\,(k-k')+\widetilde{\varepsilon}\,(k)+\widetilde{\varepsilon}\,(k')}\,\frac{C\,(k')}{\sqrt{C^2\,(k')+\xi^2\,(k')}}\,d\mathbf{k}'. \quad (8)$$

We note that this equation has a singular property: as $g^2 \to 0$, the solution C tends to zero as

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*An account of the details of diagrams for the many body problem can be found in the detailed paper of Gugenholtz.[5]

$$\exp\{-A/g^2\}, \quad A = \text{const} > 0,$$

in view of the fact that the integral on the right side of (8) remains logarithmically divergent in the vicinity of the surface $\xi(k) = 0$ if we set $C = 0$.

In such a situation, it is not difficult to obtain an asymptotic form of the solution for small g:

$$C(k) = \tilde{\omega} e^{-1/\rho} \, \frac{1}{2} \int\limits_{-1}^{+1} \frac{\omega\{k_0 \sqrt{2(1-t)}\}}{\omega\{k_0 \sqrt{2(1-t)}\} + |\xi(k)|} \, dt, \qquad (9)$$

where

$$\rho = g^2 \frac{1}{2\pi^2} \left(\frac{k^2}{d\tilde{E}(k)/dk} \right)_{k=k_0}; \qquad \tilde{E}(k_0) = \lambda,$$

$$\ln \tilde{\omega} = \int\limits_0^\infty \ln \frac{1}{2\xi} \frac{d}{d\xi} \left\{ \frac{1}{2} \int\limits_{-1}^{+1} \frac{\omega\{k_0 \sqrt{2(1-t)}\}}{\omega\{k_0 \sqrt{2(1-t)}\} + \xi} \, dt \right\}^2 d\xi. \qquad (10)$$

Taking into account the additional condition (3) and the expressions (7), (9) for u, v, it can be noted that

$$k_0 = k_F.$$

Furthermore, it is clear that the corrections to Eq. (5), obtained by replacing u_k, v_k in (5) by their "normal" values

$$\begin{aligned} u_k &= \theta_G(k) = \begin{cases} 1, & |k| > k_F \\ 0, & |k| < k_F \end{cases} \\ v_k &= \theta_F(k) = \begin{cases} 0, & |k| > k_F \\ 1, & |k| < k_F \end{cases} \end{aligned} \qquad (11)$$

will be exponentially small.

We can therefore replaced $\tilde{E}(k)$ by the corresponding formula for the normal state in Eqs. (10) without loss of accuracy, and integrate the factor

$$\frac{1}{2\pi^2} \left(\frac{k^2}{d\tilde{E}/dk} \right)_{k=k_F} = \frac{1}{V} \left\{ \frac{V}{(2\pi)^3} \frac{4\pi k^2 dk}{dE} \right\}_0$$

as the relative density dn/dE of the number of electron levels in an infinitely narrow energy band close to the Fermi surface. Then

$$\rho = g^2 dn/dE. \qquad (12)$$

We now proceed to the calculation of the energy of the ground state in the second approximation.

Of the entire H_{int}, we shall now take into account only H_1. Consequently, we get for the eigenvalue of H in the ground state

$$U - \langle C_V^* H_1 H_0^{-1} H_1 C_V \rangle =$$

$$= 2 \sum_k \{E(k) - \lambda\} v_k^2 - \frac{1}{V} \sum_{k \neq k'} g^2 \frac{\omega(k'-k)\{u_k^2 v_{k'}^2 + u_k v_k u_{k'} v_{k'}\}}{\omega(k'-k) + \varepsilon(k) + \varepsilon(k')}. \qquad (13)$$

Then, substituting the expressions we have obtained for u_k, v_k, we compute the difference ΔE between the energy of the ground state and the energy of the normal state. We have

$$\frac{\Delta E}{V} = -\frac{dn}{dE} \frac{\tilde{\omega}^2}{2} \exp\left(-\frac{2}{\rho}\right). \qquad (14)$$

It is interesting to observe that this result coincides with the result of Bardeen and his coworkers.[3] To see this we need only write the Bardeen parameters ω, V in the following fashion:

$$2\omega = \tilde{\omega}; \; V = g^2. \qquad (15)$$

We now construct, in the given approximation, the formula for the energy of the elementary excitation. For this purpose, we choose an excited state

$$C_1 = \overset{+}{\alpha}_{k0} C_V$$

and apply perturbation theory to it in the usual fashion. We obtain the following expression for the energy of an elementary excitation with momentum k:

$$E_e(k) = \varepsilon(k) - \langle C_1^* H_{\text{int}} (H_0 - \varepsilon(k))^{-1} H_{\text{int}} C_1 \rangle'_{\text{coup}},$$

for which we find

$$\begin{aligned} E_e(k) = \tilde{\varepsilon}(k) \left\{ 1 - \frac{g^2}{V} \sum_{k'} \omega(k-k') \frac{u_k^2 u_{k'}^2 + v_k^2 v_{k'}^2}{[\omega(k-k') + \varepsilon(k')]^2 - \varepsilon^2(k)} \right\} \\ + \frac{g^2}{V} 2u_k v_k \sum_{k'} \frac{\omega(k-k')(\omega(k-k') + \varepsilon(k'))}{[\omega(k-k') + \varepsilon(k')]^2 - \varepsilon^2(k)} u_{k'} v_{k'}. \end{aligned} \qquad (16)$$

Here the first term, proportional to $\tilde{\epsilon}(k)$ has no singular properties whatever and vanishes on the Fermi surface. The second term is equal to

$$\frac{g^2}{V} 2u_k v_k \sum_{k'} \frac{\omega(k-k')}{\omega(k-k') + \varepsilon(k')} u_{k'} v_{k'}$$

$$= 2u_k v_k C(k) = C(k_F) = \tilde{\omega} e^{-1/\rho}.$$

on the Fermi surface.

Thus the energies of the excited states are different from the energy of the ground state by the gap*

$$\Delta = \tilde{\omega} e^{-1/\rho}. \qquad (17)$$

We note that in the work of Bardeen and his coworkers,[3] there is an expression of the type $2\overline{\omega} \times e^{-1/\rho}$ which is interpreted there as the energy necessary to break up a "pair."

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*Here we have the fermion part of the spectrum in mind; there is, in addition, the boson part which, it appears, is not separated by a gap from the ground state. We shall not cite the corresponding formulas here since they have no relation to the theory of the superconducting state at zero temperature.

We now consider the "ground current state," i.e., the state with the lowest energy among all the possible states with given momentum **p**.

We therefore need to find the eigenvalue of H for the additional condition

$$\sum_{k,s} \mathbf{k} a_{ks}^{+} a_{ks} = \mathbf{p}.$$

Instead, making use of the usual method, we introduce (in addition to the scalar parameter λ) an additional vector parameter **u** which plays the role of an average velocity, and choose the total Hamiltonian in the form

$$H = H_{\mathrm{Fr}} - \lambda \sum_{k,s} a_{ks}^{+} a_{ks} - \sum_{k,s} (\mathbf{uk})\, a_{ks}^{+} a_{ks} =$$

$$= \sum_{k,s} \{E(k) - (\mathbf{uk}) - \lambda\}\, a_{ks}^{+} a_{ks} + \sum_{q} \omega(q)\, b_q^{+} b_q + H_{\mathrm{int}}. \tag{18}$$

The value of **u** is determined from the condition

$$\sum_{k,s} \mathbf{k}\overline{a_{ks}^{+} a_{ks}} = \mathbf{p}.$$

Since we are always dealing in our discussions only with a small region on the Fermi surface, we can for simplicity write the following

$$E(k) = \frac{k^2}{2m} + D; \quad D = E_{\mathrm{F}} - \frac{k_{\mathrm{F}}^2}{2m},$$

and then, in the final equations, take

$$m = \left(\frac{k}{dE/dk}\right)_{k=k_{\mathrm{F}}}.$$

Then

$$E(k) - (\mathbf{uk}) = E(\mathbf{k} - m\mathbf{u}) - {}^{1}/_{2}\, m\mathbf{u}^2,$$

and therefore, making the following translation in the momentum space of **k**:

$$\mathbf{k} \to \mathbf{k} + \mathbf{u}m, \quad a_{\mathbf{k}s} \to a_{\mathbf{k}+m\mathbf{u},\, s} \tag{19}$$

and putting

$$\lambda \to \lambda + m\mathbf{u}^2/2,$$

then the Hamiltonian (18) takes the form (2) and the vector **u** drops out. We again arrive at the case of a ground state with zero momentum. No new investigation is thus needed for the current state, but it will suffice only to make a transformation inverse to (19) in the formulas obtained earlier.

Thus, we can verify, for example, that the energy of the ground current state, with mean velocity **u**, differs from the energy of the ground state without current by an amount $Nm\mathbf{u}^2/2$.

The excitations are separated from the energy of the ground current state by a gap

$$\Delta_u = \Delta - \mathbf{k}_{\mathrm{F}}\mathbf{u} > \Delta - k_{\mathrm{F}}\,|\mathbf{u}|.$$

Consequently, if

$$k_{\mathrm{F}}\,|\mathbf{u}| < \Delta,$$

the current state, although it possesses an energy greater than the non-current (so long as the action of the magnetic field is not taken into account), is shown to be stable relative to the excitations.

Thus, we see that our model contains the property of superconductivity.

Let us make a few other observations.

In our method of investigation, we must take the parameter ρ to be small in order to make it possible to restrict ourselves to asymptotic approximations. However, as was shown by Tolmachev and Tiablikov[6] (with the aid of a method which does not make use of any assumption on the smallness of ρ) the sound velocity becomes imaginary for $\rho > \frac{1}{2}$, i.e., the lattice becomes unstable. In cases in which the lattice is so rigid that the electron-phonon interaction has no essential effect on the energy of the phonon, the parameter ρ must be small. Even for $\rho = {}^{1}/_{4}$, the value of $e^{-1/\rho}$ is ${}^{1}/_{55}$. This, in our opinion, explains the smallness of the value of the energy gap and therefore of the critical temperature.

We further note that if we explicitly add a term of Coulomb interaction to the Fröhlich Hamiltonian, then it will be necessary to carry out a summation of the diagrams of electron-holes of the Gelman-Bruckner type, in order to guarantee the phenomenon of screening.

By way of a preliminary estimate, we could first introduce into the Hamiltonian the Coulomb interaction in the screened form, and also treat it by means of perturbation theory. Then we would have obtained substantially the same formulas as earlier, but we would need to replace g^2 by

$$g^2 - \frac{1}{2}\int_{-1}^{1} \nu_e\left(k_{\mathrm{F}}\sqrt{2(1-t)}\right) dt,$$

where $\nu_e(\mathbf{k})$ is the Fourier transform of the screened function.

Thus it could be immediately established that the Coulomb interaction contradicts the phenomenon of superconductivity.

In conclusion I consider it my pleasant duty to thank D. N. Zubarev, V. V. Tolmachev, S. V. Tiab-

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*A NEW METHOD IN THE THEORY OF SUPERCONDUCTIVITY. II**

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The equivalence of the Bardeen Hamiltonian and the Fröhlich Hamiltonian is established in the adiabatic approximation. The energy of the ground state and of elementary excitations are calculated by means of a canonical transformation.

IN Ref. 1, Bogoliubov has shown that the property of superconductivity is possessed by a model of an electron gas in which the mutual interaction of the electrons is neglected but their interaction with the phonon field is taken into account. These results were established with the help of the Fröhlich Hamiltonian for the description of the system:[2]

$$H = H_{\mathrm{el}} + H_{\mathrm{int}} + H_{\mathrm{ph}}; \tag{1}$$

$$H_{\mathrm{el}} = \sum_{(k,\sigma)} (E(k) - \lambda)\, a_{k,\sigma}^{+} a_{k,\sigma}; \quad H_{\mathrm{ph}} = \sum_{(q)} \hbar\omega(q)\, b_q^{+} b_q; \tag{2}$$

$$H_{\mathrm{int}}^{\mathrm{Fr}} = \frac{g}{\sqrt{2V}} \sum_{(k,k',\sigma)} \hbar\omega(k-k')(a_{k',\sigma}^{+} a_{k,\sigma} b_{k-k'} + a_{k,\sigma}^{+} a_{k',\sigma} b_{k'-k}^{+}), \tag{3}$$

where E(k) is the energy of the electron; ħω(q) the energy of the phonon; k, q are the wave vectors, σ the spin variable ($\sigma = \pm \frac{1}{2}$); V the volume of the system; g the coupling constant; and λ the chemical potential. The creation and annihilation operators of electrons (a^+, a) and phonons (b^+, b) satisfy the usual commutation relations, and λ is defined by the condition

$$\sum_{k,\sigma} \overline{a_{k,\sigma}^{+} a_{k,\sigma}} = \overline{N},$$

where $\overline{N}$ is the given number of electrons.

The results of Bardeen[3] were obtained with the use of a certain equivalent Hamiltonian of the electron-electron interaction in place of the Hamiltonian (3), under not completely clear assumptions as to the suitability of the formation of electron pairs on the Fermi surface. Below, we shall show the equivalence of the Hamiltonians of Bardeen and Fröhlich, and shall establish the property of superconductivity for the Hamiltonian of Bardeen thus obtained. In the calculation, we shall make use of the method of Bogoliubov.[1]

The characteristic feature of electron-phonon interaction [Eq. (3)] is the fact that it is effective only in a thin layer at the Fermi surface, and falls off rapidly with increasing distance from it. Therefore, the principal contributions to all effects will be made by electron transitions at the Fermi surface. In this case the energy of the electron transitions can be regarded as small in comparison with the energy of the phonons $\hbar\omega$, and we have typical adiabatic coupling, wherein the frequencies of one subsystem (the electrons) are regarded as small in comparison with the frequencies of the other subsystem (phonons). We note further that $H_{\text{int}} \approx \sqrt{\hbar\omega}$ and consider H_{int} to be of first order of smallness, while H_{el} is of second order of smallness. For convenience in the intermediate calculations, we introduce the small parameter ϵ. We shall set ϵ equal to unity in the final results. Under these assumptions, the Hamiltonian of the system (1) can be written in the form:

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*The first paper of this series[1] will be denoted by I.

$$H = \varepsilon^2 H_{\text{ph}} + \varepsilon H_{\text{int}} + H_{\text{el}}. \qquad (4)$$

We make use further of the operator form of pertrubation theory.[4] For this purpose we denote by P the projection operator,* which projects the eigenfunctions C of the operator H on the subspace of eigenfunctions of the operator H_{ph}, while we denote by C_0 the eigenfunctions of this subspace ($C_0 = PC$). In this case the problem of finding the eigenfunctions and eigenvalues of the equation

$$(H - E)\,C = 0 \qquad (5)$$

reduces to the solution of the equation with a certain "deformed" operator. With accuracy up to terms in ϵ^2 inclusively (to which we also limit ourselves below) this equation has the form:

$$(E - E_0)\,C_0 = P\{\varepsilon H_{\text{int}} + \varepsilon^2 H_{\text{el}} - \varepsilon^2 (H_{\text{int}} - PH_{\text{int}}P)$$
$$(H_0 - E_0)^{-1} \times (H_{\text{int}} - PH_{\text{int}}P)\}\,PC_0, \qquad (6)$$

where E_0 is the eigenvalue of the operator H_{ph}.

We first consider the case of a phonon vacuum and set $E_0 = 0$. We also note that $PH_{\text{el}}P = H_{\text{el}}P$, since H_{el} does not act on the phonon variables and that $PH_{\text{int}}P = 0$, since H_{int} is linear in the operators b and b^+. Thus the third term in the curly brackets of (6) takes the form

$$PH_{\text{int}}\,(H_0 - E_0)^{-1}\,H_{\text{int}}P$$

$$= \frac{g^2}{2V} \sum_{\left(\substack{k_1,\, k_2,\, k_1',\, k_2',\, \sigma_1,\, \sigma_2 \\ k_1 - k_1' = k_2' - k_2 \neq 0}\right)} a^+_{k_1',\, \sigma_1} a_{k_1,\, \sigma_1} a^+_{k_2',\, \sigma_2} a_{k_2,\, \sigma_2}. \qquad (7)$$

In this expression it is easy to see that only the terms with $\sigma_1 \neq \sigma_2$ are different from zero. Putting the creation operators on the left and the annihilation operators on the right and adding the quadratic form in a^+a (isolated by this process) to the kinetic energy operator, we write down Eq. (6) in the following form

$$EC_0 = (H_0 + H^{\text{B}}_{\text{int}})\,C_0, \qquad (8)$$

where

$$H_0 = \sum_{(k,\,\sigma)} \varepsilon(k)\, a^+_{k,\sigma} a_{k,\sigma}\,;\ \varepsilon(k) = E(k) - \frac{g^2}{2V} \sum_{(k_1)} 1 - \lambda; \qquad (9)$$

$$H^{\text{B}}_{\text{int}} = \frac{g^2}{V} \sum_{\left(\substack{k_1,\, k_2,\, k_1',\, k_2' \\ k_1 - k_1' = k_2' - k_2 \neq 0}\right)} a^+_{k_1',\, 1/2}\, a^+_{k_2',\, -1/2}\, a_{k_1,\, 1/2}\, a_{k_2,\, -1/2} \qquad (10)$$

(here and below, the formal small parameter ϵ is set equal to unity). The second component in $\epsilon(k)$ takes into account the usual correction to the chemical potential.

The interaction term $H^{\text{B}}_{\text{int}}$ in Eq. (10) coincides with the corresponding expression used in the work of Bardeen, Cooper, and Schrieffer. In our derivation of Eq. (8), we have considered the energy of electron transitions to be small in comparison with the energy of the phonons $\hbar\omega$, which is valid only for transitions in a spherical shell near the Fermi surface. The breadth of this layer will evidently be of the order of some effective frequency $\hbar\tilde{\omega}$. Use of the original Hamiltonian of Fröhlich H_{int} of Eq. (3) leads to a similar result, as was shown by Bogoliubov in I.

Application of perturbation theory to the operators (8)—(10) leads to a logarithmic divergence at large distances from the Fermi surface. The reason for this appearance lies in the fact that in the derivation of (8) in the adiabatic approximation the energy of electron transitions was considered to be small in comparison with the energy of the phonons. However, the latter is only valid near the Fermi surface in a certain spherical shell of thickness of the order of the "effective phonon energy" $\hbar\tilde{\omega}$.

In the more accurate variant of the research of Bogoliubov, it is seen that the principal contribution in all quantities is given by effects which take place in this layer.

In correspondence with these considerations, we introduce a cutoff parameter into Eqs. (8)—(10) and consider the equation in the spherical shell $k_F \pm \Delta$ where k_F is the Fermi wave vector which is determined from the condition $\epsilon(k) = 0$. In view of the logarithmic singularity, the quantities ought to be only slightly sensitive to the choice of the cutoff parameter Δ.

We note that the corresponding equation in the work of Bardeen, Cooper, and Schrieffer is understood in precisely this sense.

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*The operator P is defined in the following manner: $PC = \varphi_0(\varphi_0\ C)$, where φ_0 is the eigenfunction of the operator H_{ph}; the parentheses denote scalar multiplication.

We carry out a canonical transformation[1] on the operators

$$a_{k,\,1/2} = u_k\alpha_{k,\,1} + v_k\alpha^{+}_{-k,\,0};\quad a_{k,\,-1/2} = u_k\alpha_{k,\,0} - v_k\alpha^{+}_{-k,\,1};$$

$$a^{+}_{k,\,1/2} = u_k\alpha^{+}_{k,\,1} + v_k\alpha_{-k,\,0};\quad a^{+}_{k,\,-1/2} = u_k\alpha^{+}_{k,\,0} - v_k\alpha_{-k,\,1}; \qquad (11)$$

$$u_k^2 + v_k^2 = 1;\quad (u_{-k} = u_k;\ v_{-k} = v_k), \qquad (12)$$

where α, α^{+} are new Fermi operators; u_k, v_k are the coefficients of the transformation, which will be defined below.

In the new variables, the Hamiltonian of Eq. (8) can be written in the form

$$H = \varepsilon_0 + H_0 + H_1 + H_2, \qquad (13)$$

where

$$\varepsilon_0 = 2\sum_{(k)} \varepsilon(k)\, v_k^2; \qquad (14)$$

$$H_0 = \sum \varepsilon(k)\,(u_k^2 - v_k^2)\,(\alpha^{+}_{k,\,1}\alpha_{k,\,1} + \alpha^{+}_{k,0}\,\alpha_{k,0}); \qquad (15)$$

$$H_1 = 2\sum \varepsilon(k)\, u_k v_k\,(\alpha^{+}_{k,\,1}\alpha^{+}_{-k,\,0} + \alpha_{-k,\,0}\alpha_{k,\,1}) \qquad (16)$$

$$H_2 = \frac{g^2}{V}\sum (u_{k_1'}u_{k_2'}\alpha^{+}_{k_1',\,1}\alpha^{+}_{k_2',\,0} - v_{k_1'}v_{k_2'}\alpha_{-k_1',\,0}\alpha_{-k_2',\,1}$$

$$- u_{k_1'}v_{k_2'}\alpha^{+}_{k_1',\,1}\alpha_{-k_2',1} + v_{k_1'}u_{k_2'}\alpha_{-k_1',\,0}\alpha^{+}_{k_2',\,0})$$

$$(u_{k_1}u_{k_2}\alpha_{k_1,\,1}\alpha_{k_2,\,0} - v_{k_1}v_{k_2}\alpha^{+}_{-k_1,\,0}\alpha^{+}_{-k_2,\,1}$$

$$- u_{k_1}v_{k_2}\alpha_{k_1,\,1}\alpha^{+}_{-k_2,\,1} + v_{k_1}u_{k_2}\alpha^{+}_{-k_1,\,0}\alpha_{k_2,\,0}). \qquad (17)$$

We shall further regard H_0 as the operator of zeroth approximation and H_1, H_2 as perturbations.

It was shown in I that, in line with perturbation theory, there will be dangerous terms corresponding to the creation of two particles; it is shown there that this is connected with the presence of logarithmic singularities on the Fermi surface.

The situation is very much the same for the Hamiltonian (13). To remove the difficulties with divergence in first order in g^2, we sum over all diagrams leading from the vacuum to the two-particle states, and so choose the functions u_k, v_k that they compensate one another. One such diagram is in the Hamiltonian H_1:

$$2\sum \varepsilon(k)\, u_k v_k \alpha^{+}_{k,\,1}\alpha^{+}_{-k,\,0}$$

and two in the Hamiltonian H_2:

$$\frac{g^2}{V}\sum_{\substack{k_1,\,k_2,\,k_1',\,k_2' \\ k_1 - k_1' = k_2' - k_2 \neq 0}} (-v_{k_1'}u_{k_2'}v_{k_1}v_{k_2}\delta_{-k_1',\,k_2'}\alpha^{+}_{-k_1,\,0}\alpha^{+}_{-k_2,\,1}$$

$$- u_{k_1'}u_{k_2'}u_{k_1}v_{k_2}\alpha^{+}_{k_1',\,1}\alpha^{+}_{k_2',\,0}\delta_{k_1,\,-k_2})$$

$$= -\frac{g^2}{V}\sum_{(k_1)} u_{k_1}v_{k_1}\sum_{(k)}(u_k^2 - v_k^2)\,\alpha^{+}_{k,\,1}\alpha^{+}_{-k,\,0}.$$

The vanishing of the coefficients of $\alpha^{+}_{k,1}\alpha^{+}_{-k,0}$ leads to a system of equations for the functions u_k, v_k:

$$2\varepsilon(k)\, u_k v_k = (u_k^2 - v_k^2)\,\frac{g^2}{V}\sum_{(k_1)} u_{k_1}v_{k_1}. \qquad (18)$$

We note, furthermore, that the set (18) has a trivial solution $u_k v_k = 0$ and hence

$$u_k^2 = \begin{cases} 0, & |k| < k_F, \\ 1, & |k| > k_F; \end{cases}\qquad v_k^2 = \begin{cases} 1, & |k| < k_F, \\ 0, & |k| > k_F. \end{cases} \qquad (19)$$

This solution, as is seen from what follows, corresponds to the normal (non-superconducting) state of the system.

In order to find the nontrivial solution, we introduce the notation:

$$C = (g^2/V)\sum u_k v_k. \qquad (20)$$

Then, making use of the normalization condition (12) for the functions u_k and v_k, it is not difficult to solve the set (18) relative to u_k^2 and v_k^2:

$$u_k^2 = \frac{1}{2}\left\{1 + \frac{\varepsilon(k)}{\sqrt{C^2 + \varepsilon^2(k)}}\right\};$$

$$v_k^2 = \frac{1}{2}\left\{1 - \frac{\varepsilon(k)}{\sqrt{C^2 + \varepsilon^2(k)}}\right\}. \qquad (21)$$

Substituting the results found for $u_k v_k$ in (20), we get an equation for the constant C:

$$1 = (g^2/2V)\sum_{(k)} [C^2 + \varepsilon^2(k)]^{-1/2}. \qquad (22)$$

Transforming from sums to integrals in (22), and taking it into consideration that the range of integration in k lies in the interval $(k_F - \Delta,\ k_F + \Delta)$, we can obtain for C the following asymptotic solution:

$$C = 2\tilde{\Delta}e^{-1/\rho};\quad \rho = 2g^2k_F^2/(2\pi)^2\,\varepsilon'(k_F);\quad \tilde{\Delta} = \varepsilon'(k_F)\,\Delta. \qquad (23)$$

We now compute the energy of the elementary excitations in first approximation in g^2:

$$\Omega(k,1) = \varepsilon(k)\,(u_k^2 - v_k^2) + 2u_k v_k\,\frac{g^2}{V}\sum_{(k_1)} u_{k_1}v_{k_1},$$

or, utilizing Eqs. (20) and (21)*

$$\Omega(k,\gamma) = \sqrt{C^2 + \varepsilon^2(k)}, \qquad (24)$$

where the constant C is determined by Eq. (23).

For the normal state, obviously,

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*Equation (24) was obtained by Bardeen, Cooper and Schrieffer, as pointed out in the paper of Bernardes.[5]

$$\Omega(k,\gamma)=\varepsilon(k). \qquad (25)$$

We see from (24) that, in contrast to the normal state, the energy of the elementary excitations differs from the ground state by the gap

$$\Delta\Omega=C. \qquad (26)$$

Just as in I, we can consider the current state, i.e., the state with the total momentum of the system different from zero. In this case we can show that for sufficiently small velocities u, the value of the gap falls off proportional to u.

Proceeding to the computation of the addition to the energy of the ground state, we note that the contribution in it amounts to only one term. As a result, we obtain

$$\Delta\varepsilon_0=-\frac{g^2}{V}\Big(\sum_{(k)}u_kv_k\Big)^2=-\frac{V}{g^2}C^2 \qquad (27)$$

(for the normal state, $C = 0$ and $\Delta\epsilon_0 = 0$).

We now show that the superconducting state is energetically more advantageous than the normal. Making use of Eqs. (14), (27), and (31), we obtain for the difference of the lowest energy levels of the superconducting and normal states

$$\Delta E=\varepsilon_0+\Delta\varepsilon_0-\sum\varepsilon(k)\Big\{1-\frac{\varepsilon(k)}{|\varepsilon(k)|}\Big\}=$$
$$=\sum\Big\{|\varepsilon(k)|-\frac{\varepsilon^2(k)}{\sqrt{C^2+\varepsilon^2(k)}}\Big\}-\frac{V}{g^2}C^2. \qquad (28)$$

Hence, with the same degree of accuracy as in the asymptotic solution for C, we finally obtain

$$\Delta E=-V\frac{k_F^2\,(2\widetilde{\Delta})^2}{(2\pi)^2\,\varepsilon'(k_F)}e^{-2/\rho}. \qquad (29)$$

We can interpret the quantity

$$\frac{2k_F^2}{(2\pi)^2\,\varepsilon'(k_F)}=\frac{1}{V}\Big\{\frac{V}{(2\pi)^3}\frac{4\pi k^2dk}{dE}\Big\}_{k=k_F}$$

as the relative density of electron levels, dn/dE. Then (29) is rewritten in the following form:

$$\frac{1}{V}\Delta E=-\frac{dn}{dE}\frac{(2\widetilde{\Delta})}{2}e^{-2/\rho};\quad \rho=g^2\frac{dn}{dE}\cdot \qquad (30)$$

Identifying the thickness of the layer $2\widetilde{\Delta}$ with the quantity $\widetilde{\omega}$ from Eq. (14) of I, we obtain agreement with all the results obtained by Bogoliubov with the use of the Hamiltonian of Fröhlich. If we carry out the substitutions $\widetilde{\Delta}\to\omega$, $g^2\to V$, $dn/dE = N$ in (30), then clear agreement of the results is obtained with the results of Bardeen et al.[3] The minimum energy necessary to break up the pair calculated in Ref. 4 is evidently equal to twice the energy of the gap in the spectrum of elementary excitations (26). In the same way, complete correspondence is obtained between the Hamiltonians of Fröhlich and Bardeen and the results obtained with their help.

It should be pointed out that Eqs. (26) and (30) are only slightly sensitive to change in the form of the interaction assumed by us. Thus, if we replace the constant g^2 by the quantity $g^2(k_1-k_2)$, concentrated about the Fermi surface, then Eqs. (26), (30) would be preserved with this one difference, that g^2 would be replaced by the mean value of $g^2(k_1-k_2)$, and $\widetilde{\Delta}$ would be replaced by some mean width $\widetilde{\Delta}(k_1-k_2)$. On the other hand, the Bose part of the spectrum can depend on the detailed form of the function $g(k_1-k_2)$. We intend to consider this dependence in a subsequent paper.

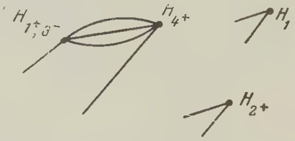

FIG. 1

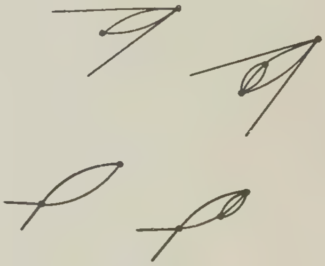

FIG. 2

The results worked out above were obtained in first order perturbation theory. It is not difficult to establish the fact that the compensation of diagrams in second order (g^4) does not change the results. Actually, there are terms in the perturbation operator H_2 [Eq. (17)] which describe, in their effect on the vacuum, the creation of four particles — H_4+, the creation of a single particle and the annihilation of three particles — $H_1+,_3-$ etc. In second order perturbation theory, the contribution to the diagrams with two particles as a result give terms of the form

$$H_{1^+,3^-}(H_0-E_0)^{-1}H_{4^+}, \qquad (31)$$

which are drawn in Fig. 1.

We now require that they be compensated in the sum with the diagrams obtained from H_1 and H_2 in first order, pictured in this same figure.

We note that the diagrams of the form shown in Fig. 2 are compensated automatically in this case by virtue of the compensation rule.

The operators H_4+ and $H_1+,_3-$ have the form

$$H_{1+} = -\frac{g^2}{V} \sum_{\substack{k_1,\, k_2,\, k_1',\, k_2' \\ k_1 - k_1' = k_2' - k_2' \neq 0}} u_{k_1'} u_{k_2'} v_{k_1} v_{k_2} \alpha^{+}_{k_1' 1} \alpha^{+}_{k_2' 0} \alpha^{+}_{-k_1 0} \alpha^{+}_{-k_2 0}; \qquad (32)$$

$$H_{1+,\,3-} = -\frac{g^2}{V} \sum_{\substack{k_1,\, k_2,\, k_1',\, k_2' \\ k_1 - k_1' = k_2' - k_2 \neq 0}} (v_{k_1'} v_{k_2'} u_{k_1} v_{k_2} - u_{k_1'} u_{k_2'} v_{k_1} u_{k_2}) \times \alpha^{+}_{-k_2 1} \alpha_{-k_2' 1} \alpha_{k_1 1} \alpha_{-k_1' 0}$$

$$- \frac{g^2}{V} \sum_{\substack{k_1,\, k_2,\, k_1',\, k_2' \\ k_1 - k_1' = k_2' - k_2 \neq 0}} (v_{k_1'} v_{k_2'} v_{k_1} u_{k_2} - u_{k_1'} u_{k_2'} u_{k_1} v_{k_2}) \times \alpha^{+}_{-k_1 0} \alpha_{-k_1' 0} \alpha_{-k_2' 1} \alpha_{k_2 0}. \qquad (33)$$

Since the second term is obtained from the first by the substitution it is evident that it only doubles the effect of first term.

As a result, we obtain for u_k, v_k the following set of equations:

$$2 \Bigg\{ \varepsilon(k) - \left(\frac{g^2}{V}\right)^2 \sum_{\substack{k_1,\, k_2,\, k_3 \\ k_2 - k_1 + k_3 = k}} \frac{\{u_{k_1}^2 v_{k_2}^2 - (u_{k_1} v_{k_1})(u_{k_2} v_{k_2})\} (u_{k_3}^2 - v_{k_3}^2)}{\varepsilon(k) + \varepsilon(k_1) + \varepsilon(k_2) + \varepsilon(k_3)} \Bigg\}$$

$$= (u_k^2 - v_k^2) \Bigg\{ \frac{g^2}{V} \sum_{(k_1)} u_{k_1} v_{k_1} + \left(\frac{g^2}{V}\right)^2 \sum_{\substack{k_1,\, k_2,\, k_3 \\ k_2 - k_1 + k_3 = k}} \frac{(u_{k_1} v_{k_1}) \{(u_{k_2} v_{k_2})(u_{k_3} v_{k_3}) - u_{k_2}^2 v_{k_3}^2\}}{\varepsilon(k) + \varepsilon(k_1) + \varepsilon(k_2) + \varepsilon(k_3)} \Bigg\}. \qquad (34)$$

In view of the fact that the additional terms of fourth order in g contain products of uv with the same index, which give an exponentially small contribution, then, as one can see, they do not change the asymptotic solution found earlier. We further note that terms of fourth order were considered for the model Hamiltonian (10), which is itself obtained with accuracy to terms of order g^2. Therefore, if we attempt to obtain the exact corrections from terms of order g^4, it is necessary first to improve the model Hamiltonian (10), extending the method of projection up to terms of fourth order.

In conclusion, the authors take the opportunity to express their gratitude to N. N. Bogoliubov, as well as to D. N. Zubarev and Iu. A. Tserkovnikov for discussion of the work.

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A NEW METHOD IN THE THEORY OF SUPERCONDUCTIVITY. III

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The Hamiltonian of Bardeen is analyzed in this paper and the method of summation of the principal diagrams is applied. It is shown that the same results are obtain in this way as in Refs. 1 and 2.

RECENTLY great successes have been attained in the solution of problems of statistical physics by means of summation of the principal diagrams.

In the present paper we shall show that in the theory of superconductivity, we can also obtain (by this method) those results which were found in previous researches[1,2] with the aid of a canonical transformation and the principle of compensation of diagrams with "dangerous" energy denominators.

As has been shown by Tolmachev and Tiablikov,[2] we can consider the Hamiltonian of Bardeen in place of that of Fröhlich, since they both, up to a known degree, give equivalent results for the effect of electron-phonon interaction on the dynamics of electrons close to the Fermi surface. In our case the Bardeen Hamiltonian is considerably simpler.

Therefore, for a more graphic description and to establish the connection with the ideas of the work of Bardeen, Cooper, and Schrieffer,[3] we shall start out from the Hamiltonian of Bardeen:

$$H_{\mathrm{B}} = \sum_{k,\,s} E(k)\, a_{ks}^{+} a_{ks}$$

$$- \frac{I}{V} \sum_{(k_1\, k_2,\, k_1',\, k_2')} a^{+}_{k_1,\, -1/2}\, a^{+}_{k_2,\, 1/2}\, a_{k_2'\, 1/2}\, a_{k_1',\, -1/2}\, \theta(k_1)\, \theta(k_2)$$

$$\times\, \theta(k_1')\, \theta(k_2')\, \Delta(k_1 + k_2 - k_1' - k_2'),$$

where

$$\theta(k) = \begin{cases} 1, & E(k_{\mathrm{F}}) - \omega < E(k) < E(k_{\mathrm{F}}) + \omega \\ 0, & |E(k) - E(k_{\mathrm{F}})| > \omega \end{cases},$$

$$\Delta(k) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0 \end{cases}$$

and where E(k) is the radially symmetric function representing the energy of an electron of momentum k; I and ω are the Bardeen parameters. In the Fröhlich model, we must set[2]

$$I = g^2, \quad \omega = \tilde{\omega}/2.$$

We shall take into account the value N of the total number of electrons by means of the chemical potential λ, for which purpose we add the term $-\lambda N$ to H_{B}. Then we obtain the Hamiltonian

$$H = H_0 + H_{\mathrm{int}},$$

$$H_0 = \sum_{k,\, s} \{E(k) - \lambda\}\, a_{ks}^{+} a_{ks},$$

$$H_{\mathrm{int}} = -\frac{I}{V} \sum_{(k_1,\, k_2,\, k_1',\, k_2')} a^{+}_{k_1,\, -1/2}\, a^{+}_{k_2,\, 1/2}\, a_{k_2',\, 1/2}\, a_{k_1',\, -1/2}$$

$$\times\, \theta(k_1)\, \theta(k_2)\, \theta(k_1')\, \theta(k_2')\, \Delta(k_1 + k_2 - k_1' - k_2'),$$

for which we shall also consider the question of the summation of the principal diagrams.

Since the interaction is effective only in a small region of the Fermi sphere and only between particles (electrons or holes) with oppositely directed spins, we see that a very important role will be played by diagrams of the type shown in Fig. 1. These diagrams were constructed from an "irreducible complex" (see Fig. 2), consisting of a pair of particles with momenta ± k and spins $\pm \frac{1}{2}$.

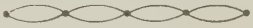

FIG. 1

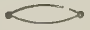

FIG. 2

To sum the diagrams, we make use of the method of approximate second quantization, i.e., we construct a simplified Hamiltonian, for which the diagrams will only be of that class which we desire to sum, and furthermore, with the same contribution which is made in the present Hamiltonian.*

Since the complexes of pairs of particles $(\pm k, \pm\frac{1}{2})$ are not broken up in the diagrams considered by us, it is natural to compare their quantum amplitudes b_k, b_k^+ with the commutation relations

$$[b_k, b_{k'}] = 0, \quad [b_k^+, b_{k'}^+] = 0, \quad [b_k^+, b_{k'}] = 0; \quad k \neq k'. \quad (2)$$

Furthermore, since there do not exist several pairs with the same value of k, we must have

$$b_k^2 = 0, \quad b_k^{+2} = 0, \quad b_k b_k^+ + b_k^+ b_k = 1. \quad (3)$$

We note further that the eigenenergy of the complex will be

$$2(E(k) - \lambda)\, b_k^+ b_k$$

and that the matrix element of the Hamiltonian (1) for the transition $k \rightarrow k'$ will be proportional to $-I/V$.

From these considerations, we obtain a simplified Hamiltonian of the form

$$H = H_0 + H_{\text{int}}, \quad H_0 = \sum_k 2(E(k) - \lambda)\, b_k^+ b_k,$$

$$H_{\text{int}} = -\frac{I}{V} \sum_{(k \neq k')} b_k^+ b_{k'} \theta(k)\, \theta(k'), \quad (4)$$

which contains the operators b_k, b_k^+ [with the commutation relations (2), (3)] which we shall call the Pauli operators.

Taking expressions of arbitrary order

$$H_{\text{int}} (H_0 - E)^{-1} H_{\text{int}} \ldots (H_0 - E)^{-1} H_{\text{int}},$$

it is now easy to verify directly that the sum of contributions from the diagrams of the type considered for the Hamiltonian (1) will be equal to the sum of contributions of all diagrams for the simplified Hamiltonian (4).

Thus, the problem of the summation of a special class of diagrams for the Hamiltonian (1) is shown to be equivalent to the problem of the model of the dynamical system that is characterized by the Hamiltonian (4).

We proceed to the construction of an asymptotically exact solution of this latter problem, neglecting only the quantities which vanish in the limit: $V \rightarrow \infty$.

We shall distinguish between pairs of electrons and pairs of holes, for which purpose we introduce the new Pauli operators, setting

$$\beta_k = b_k^+, \quad E(k) < \lambda,$$
$$\beta_k = b_k, \quad E(k) > \lambda.$$

We get

$$H = U + 2 \sum_k |E(k) - \lambda|\, \beta_k^+ \beta_k - \frac{I}{V} \sum_{k \neq k'} \theta(k)\, \theta(k') \{\theta_G(k)\, \beta_k^+ + \theta_F(k)\, \beta_k\} \{\theta_G(k')\, \beta_{k'} + \theta_F(k')\, \beta_{k'}^+\}, \quad (5)$$

where

$$\theta_F(k) = \begin{cases} 1, & E(k) < \lambda \\ 0, & E(k) > \lambda \end{cases}$$

$$U = 2 \sum_k \{E(k) - \lambda\}\, \theta_F(k), \quad \theta_G(k) + \theta_F(k) = 1.$$

Let us consider the wave function C for which all the filling factors

$$n_k = \beta_k^+ \beta_k$$

are equal to zero. Then

$$\beta_k C = 0.$$

We shall show that this wave function is an asymptotically exact eigenfunction of the Hamiltonian H, giving it the value U.

In fact, we have

$$H = H' + H'' + U,$$

$$H' = 2 \sum_k |E(k) - \lambda|\, \beta_k^+ \beta_k - \frac{I}{V} \sum_{k \neq k'} \theta(k)\, \theta(k') \{\theta_G(k)\, \beta_k^+ + \theta_F(k)\, \beta_k\}\, \theta_G(k')\, \beta_{k'} - \frac{I}{V} \sum_{k \neq k'} \theta(k)\, \theta(k')\, \theta_F(k)\, \theta_F(k')\, \beta_{k'}^+ \beta_k,$$

$$H'' = -\frac{I}{V} \sum_{k \neq k'} \theta(k)\, \theta(k')\, \theta_G(k)\, \theta_F(k')\, \beta_k^+ \beta_{k'}^+.$$

But, obviously,

$$H'C = 0.$$

On the other hand,

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*We emphasize that the new meaning of "summation" introduced here cannot be taken in the universally accepted sense. Strictly speaking, we do not sum here a series of terms of a given class, but we compare the Hamiltonian for which the expansion of perturbation theory exactly coincides with the given series. From the mathematical viewpoint, we are dealing here with concepts close to those of the theory of quasi-analytic functions.

$$\langle C^* | H'' |^2 C\rangle = \langle C^* \overset{+}{H}{}'' H'' C\rangle =$$
$$= \frac{I^2}{V^2} \sum_{k \neq k'} \theta(k)\,\theta(k')\,\theta_G(k)\,\theta_F(k') < \text{const when } V \to \infty.$$

But, in the limit $V \to \infty$, H must be proportional to V, while $|\overline{H''}|^2$, as we have noted, remains finite. Therefore, in fact, C is an asymptotically exact eigenfunction of H, giving it the value U.

We have also

$$\overline{N} = \langle C^* N C\rangle = \sum_{E(k) < \lambda} 2.$$

Equating this expression to the total number of electrons in the Fermi sphere

$$\sum_{E(k) < E_F} 2,$$

we see that

$$\lambda = E_F = E(k_F).$$

We now analyze the problem of the stability of the state C. We consider first the case in which

$$I < 0. \tag{6}$$

We supplement the double sum in (5) with terms for which $k = k'$, terms that make no contribution in the transition to the limit $V \to \infty$. We then note that $H - U$ is essentially a positive form. The value U will consequently be a minimum and the state C will be stable from the same considerations.

The situation will be different in the case

$$I > 0. \tag{7}$$

We note that since all the filling facts $n_k = \beta_k^+ \beta_k$ in the state C will be equal to zero, we can, upon computation of the energy of the elementary excitations, consider the Pauli operators β, β^+ to be Bose in character.

There then remains only the diagonalization to quadratic form of the operators β, β^+ which represent $H - U$ of (5). This diagonalization can be achieved, for example, with the aid of a method set forth in our monograph.[4]

For the determination of the energy E of the elementary excitation, we get the following secular equation:

$$1 = \frac{I}{V} \sum_k \theta(k) \left\{ \frac{\theta_F(k)}{\varepsilon_k - E} + \frac{\theta_G(k)}{\varepsilon_k + E} \right\},$$
$$\varepsilon_k = 2\,|E(k) - E(k_F)|,$$

whence, upon simplifying, we get

$$1 = \frac{I}{V} \sum_{(E_F < E(k) < E_F + \omega)} \frac{2\varepsilon_k}{\varepsilon_k^2 - E^2},$$

or

$$1 = \frac{\rho}{2} \int_0^\omega k^2 \frac{dk}{dE} \frac{2z\,dz}{z^2 - E^2/4},$$

where

$$\rho = \frac{I}{2\pi^2} \left(k^2 \frac{dk}{dE}\right)_{k=k_F}. \tag{8}$$

As is seen, this equation, in the case (7) under consideration, always has a negative root for E^2. Consequently we obtain a purely imaginary value for the energy E:

$$E \sim \pm i 2\omega e^{-1/\rho}. \tag{9}$$

Thus the state C is found to be unstable.

In order to find the stable ground state with minimum energy, we introduce the new Pauli amplitudes β_k, β_k^+ in non-trivial fashion, as earlier, with the aid of the relations

$$\begin{aligned} \beta_k &= u_k v_k (2 b_k^+ b_k - 1) + u_k^2 b_k - v_k^2 b_k^+, \\ \beta_k^+ &= u_k v_k (2 b_k^+ b_k - 1) - v_k^2 b_k + u_k^2 b_k^+, \end{aligned} \tag{10}$$

where u_k and v_k are real numbers satisfying the relation

$$u_k^2 + v_k^2 = 1. \tag{11}$$

It is not difficult to note that the amplitudes (10) actually satisfy all the commutation relations of the Pauli operators.

Turning to the transformation (10), we find:

$$\begin{aligned} b_k &= u_k v_k (1 - 2\beta_k^+ \beta_k) + u_k^2 \beta - v_k^2 \beta_k^+, \\ b_k^+ &= u_k v_k (1 - 2\beta_k^+ \beta_k) - v_k^2 \beta_k + u_k^2 \beta_k^+, \\ b_k^+ b_k &= v_k^2 + (u_k^2 - v_k^2)\,\beta_k^+ \beta_k + u_k v_k (\beta_k + \beta_k^+). \end{aligned} \tag{12}$$

Substituting these expression in the Hamiltonian (4), we find:

$$H = U + \sum \Big\{ 2(E(k) - \lambda)\, u_k v_k$$
$$- \frac{I}{V} \theta(k) (u_k^2 - v_k^2) \sum_{k'} u_{k'} v_{k'} \theta(k') \Big\} (\beta_k + \beta_k^+)$$
$$+ \sum_k 2E_e(k)\, \beta_k^+ \beta_k - \frac{I}{V} \sum_{k_1 \neq k_2} \{u_{k_1}^2 \beta_{k_1}^+ - v_{k_1}^2 \beta_{k_1} - 2u_{k_1} v_{k_1} \beta_{k_1}^+ \beta_{k_1}\}$$
$$\times \{u_{k_2}^2 \beta_{k_2} - v_{k_2}^2 \beta_{k_2}^+ - 2u_{k_2} v_{k_2} \beta_{k_2}^+ \beta_{k_2}\}\, \theta(k_1)\, \theta(k_2),$$

where

$$U = \sum_k 2(E(k) - \lambda)\, v_k^2 - \frac{I}{V} \sum_{(k_1, k_2)} \theta(k_1)\, \theta(k_2)\, u_{k_1} v_{k_1} u_{k_2} v_{k_2};$$

$$E_e(k) = (E(k) - \lambda)(u_k^2 - v_k^2) + \theta(k)\, u_k v_k \frac{2I}{V} \sum_{k'} \theta(k')\, u_{k'} v_{k'}. \tag{15}$$

We let the coefficients for $(\beta_k + \beta_k^+)$ in Eq. (13) vanish, and get

$$2(E(k) - \lambda)\, u_k v_k - \frac{I}{V}\, \theta(k)\, (u_k^2 - v_k^2) \sum_{k'} \theta(k')\, u_{k'} v_{k'} = 0, \tag{16}$$

which was found in Ref. 2 with the help of the principle of compensation of dangerous diagrams.

Noting that $\lambda = E(k_F)$ (with the accuracy required here), we have, just as in Ref. 2:

$$\begin{gathered} u^2(k) = \frac{1}{2}\left\{1 + \frac{E(k) - E(k_F)}{\sqrt{(E(k) - E(k_F))^2 + \theta(k)\, C^2}}\right\}, \\ v^2(k) = \frac{1}{2}\left\{1 + \frac{E(k_F) - E(k)}{\sqrt{(E(k) - E(k_F))^2 + \theta(k)\, C^2}}\right\}, \\ C = 2\omega e^{-1/\rho}, \\ E_e(k) = \sqrt{(E(k) - E(k_F))^2 + \theta(k)\, C^2}. \end{gathered} \tag{17}$$

The Hamiltonian (13) can now be written in the form

$$H = U + H_0 + H' + H''$$

where

$$H_0 = \sum_k 2E_e(k)\, \beta_k^+ \beta_k,$$

$$\begin{aligned} H' = -\frac{I}{V} \sum_{(k_1 \neq k_2)} \theta(k_1)\, \theta(k_2)\, \{&u_{k_1}^2 \beta_{k_1}^+ \\ &- v_{k_1}^2 \beta_{k_1}\}\, \{u_{k_2}^2 \beta_{k_2} - v_{k_2}^2 \beta_{k_2}^+\}, \end{aligned} \tag{18}$$

$$\begin{aligned} H'' = \frac{2I}{V} \sum_{(k_1 \neq k_2)} \theta(k_1)\, \theta(k_2)\, \{&u_{k_2}^2 \beta_{k_2} - v_{k_2}^2 \beta_{k_2}^+ \\ &- 2u_{k_2} v_{k_2} \beta_{k_2}^+ \beta_{k_2}\}\, u_{k_1} v_{k_1} \beta_{k_1}^+ \beta_{k_1} \\ + \frac{2I}{V} \sum_{(k_1 \neq k_2)} \theta(k_1)\, \theta(k_2)\, \{&u_{k_1}^2 \beta_{k_1}^+ - v_{k_1}^2 \beta \\ &- 2u_{k.} v_{k_1} \beta_{k_1}^+ \beta_{k_1}\}\, u_{k_2} v_{k_2} \beta_{k_2}^+ \beta_{k_2}. \end{aligned}$$

We select the wave function C for which all the filling factors

$$\nu_k = \beta_k^+ \beta_k$$

are zero. We shall show, as before, that, with accuracy to quantities which vanish in the limit $V \to \infty$, C is an eigenfunction of the Hamiltonian H, giving it a value U. We actually have

$$(H_0 + H'')\, C = 0$$

and

$$\langle C^* \,|\, H' \,|^2\, C \rangle = \frac{I^2}{V^2} \sum \theta(k_1)\, \theta(k_2)\, \{u_{k_1}^4 v_{k_2}^4 + u_{k_1}^2 v_{k_1}^2 u_{k_2}^2 v_{k_2}^2\}$$

$$< \text{const}, \quad V \to \infty.$$

We now proceed to the consideration of elementary excitations. Since all the filling factors ν_k in the state C are equal to zero, we can consider the Pauli operators to be Bose operators in the calculation of the energy of the elementary excitations. Therefore, in the equation for the Hamiltonian (18) we can limit ourselves to the quadratic form $H_0 + H'$.

Carrying out the diagonalization by the method described earlier,[4] we obtain a set of linear equations:

$$\begin{aligned} (E - 2E_e(k))\, \varphi_k = {}& \frac{I\theta(k)}{V}\, u_k^2 \sum_{k'} \{u_{k'}^2 \varphi_{k'} - v_{k'}^2 \chi_{k'}\}\, \theta(k') - \\ & - \frac{I\theta(k)}{V}\, v_k^2 \sum_{k'} \{u_{k'}^2 \chi_{k'} - v_{k'}^2 \varphi_{k'}\}\, \theta(k'), \\ -(E + 2E_e(k))\, \chi_k = {}& u_k^2 \frac{I}{V}\, \theta(k) \sum_{k'} \{u_{k'}^2 \chi_{k'} - v_{k'}^2 \varphi_{k'}\}\, \theta(k') \\ & - \frac{I}{V}\, \theta(k)\, v_k^2 \sum_{k'} \{u_{k'}^2 \varphi_{k'} - v_{k'}^2 \chi_{k'}\}\, \theta(k') \end{aligned} \tag{19}$$

with the normalization condition

$$\sum_k \{|\varphi_k|^2 - |\chi_k|^2\} = 1. \tag{20}$$

Then we obtain the secular equation:

$$\begin{gathered} \left\{1 + \frac{I}{V} \sum \left(\frac{u_k^4}{2E_e + E} + \frac{v_k^4}{2E_e - E}\right) \theta(k)\right\} \\ \times \left\{1 + \frac{I}{V} \sum \left(\frac{v_k^4}{2E_e + E} + \frac{u_k^4}{2E_e - E}\right) \theta(k)\right\} \\ - \left\{\frac{I}{V} \sum \theta(k)\, u_k^2 v_k^2 \left(\frac{1}{2E_e + E} + \frac{1}{2E_e - E}\right)\right\}^2 = 0. \end{gathered} \tag{21}$$

It is easy to see that for

$$|E| < 2 \min E_e(k) = 2E_e(k_F)$$

this equation has no solution, since the subtrahend in Eq. (21) is then less than the minuend.

For

$$|E| > 2 \min E_e(k)$$

we have a continuous spectrum

$$E = \pm 2E_e(k) + O(1/V), \quad O(1/V) \to 0 \text{ when } V \to \infty.$$

As is seen from (19), the minus sign does not agree with the normalization condition (20).

Thus all the E are positive (this can be seen directly from the fact that the quadratic form under consideration is positive definite) and are separated from zero by the gap

$$E = 2E_e(k) \geqslant 2E_e(k_F) = 2C = 4\omega e^{-1/\rho}. \qquad (22)$$

Here again we obtain the results of Bardeen as in the previous papers.[1,2]

Since we have confined ourselves only to diagrams consisting of pairs, we cannot decide directly from (22) that the excitation with energy $2E_e(k)$ [Eq. (22)] consists indeed of two excitations of the Fermi type, which was shown in Ref. 1.

As we see, the method of summation of diagrams is shown to be quite lucid and permits us to establish the connection with the ideas of the work of Bardeen, Cooper, and Schrieffer.

However, in our opinion, the method of canonical transformation is more flexible, allowing us easily to obtain the higher approximations. Moreover, it achieves various generalizations, for example, in the calculation of thermodynamic quantities.

In conclusion, I should thank D. N. Zubarev, V. V. Tolmachev, S. V. Tiablikov, and Iu. A. Tserkovnikov for their valued discussion.

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BREMSSTRAHLUNG OF PI MESONS AND PRODUCTION OF PI-MESON PAIRS BY GAMMA QUANTA IN COLLISION WITH NONSPHERICAL NUCLEI

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The effective cross-sections are calculated for a number of radiative processes occurring in the interaction of high-energy π mesons with nonspherical nuclei. The nonspherical shape of the nuclei leads to a change of the angular distributions and the appearance in the cross-sections of factors that depend only on the geometrical shape of the nuclei.

In papers by Landau and Pomeranchuk,[1] Pomeranchuk,[2] and Vdovin[3] treatments have been given of the processes of bremsstrahlung in the interaction of π mesons with nuclei, production of π-meson pairs from nuclei by γ quanta, and production of nuclear stars by γ quanta, for very large π-meson energies E and γ-quantum energies ω ($E \gg \mu$; $\omega \gg \mu$, where μ is the mass of the π meson; we set $\hbar = c = 1$ throughout). A peculiarity of these processes at such energies is that very large distances from the nucleus ($r_{eff} \sim E/\mu^2 \gg R$) contribute to the matrix elements that give the probabilities of the processes, and one can use in the calculation the asymptotic form of the π-meson wave functions outside the region of their interaction with the nucleus. At large energies one can take these functions to be diffraction functions. The functions used in the papers mentioned are those of the diffraction by a black or a gray sphere.

As is well known, the equilibrium shape of a heavy nucleus is not that of a sphere, but that of an ellipsoid of revolution. The change of the shape and the related appearance of a spectrum of rotational levels causes considerable change in the diffraction solution, as has been shown by Drozdov[4] for the scattering of fast neutrons by nuclei.

In the present paper we find the changes made by the nonspherical shape of the nucleus in radiative processes occurring in the collisions of π mesons with nuclei and in the production of π-meson pairs by γ quanta. We use in this work as a model of the nucleus an ellipsoid of revolution which is black relative to the π mesons.

1. EMISSION OF A GAMMA QUANTUM BY A PI MESON

The radiation in question consists of radiation emitted when the π meson is scattered and radiation emitted during the absorption of the meson (stopping radiation). As has been shown by Landau and Pomeranchuk,[1] both parts of the radiation can be obtained from the wave equation for the π meson outside the nucleus

$$(\partial^2/\partial t^2-\Delta+\mu^2)\psi=-2ie(\mathbf{A}\nabla)\psi, \qquad (1)$$

$$\mathbf{A}=\sum_{\omega\mathrm{j}}\sqrt{\frac{2\pi}{\omega}}\,\mathbf{j}\,(a_{\omega\mathrm{j}}e^{i(\omega\mathbf{r}-\omega t)}+a^{+}_{\omega\mathrm{j}}e^{-i(\omega\mathbf{r}-\omega t)}).$$

If we regard the interaction with the radiation field as a perturbation, we can replace ψ in the right member of Eq. (1) by the π-meson wave function $\Phi_{\mathbf{p}_1}$ in zeroth approximation with respect to e. Then the solution of Eq. (1) that corresponds to the emission of a photon with momentum ω and polarization j has the form

$$\Phi_{\mathbf{p}_2}=\int G(\mathbf{r},\mathbf{r}')\frac{2e}{i}\sqrt{\frac{2\pi}{\omega}}(\mathbf{j}\nabla\Phi_{\mathbf{p}_1})e^{-i\omega\mathbf{r}'}d\mathbf{r}', \qquad (2)$$

$$p_2^2=(E_1-\omega)^2-\mu^2,$$

where $G(\mathbf{r},\mathbf{r}')$ is the Green's function of the wave equation for the π meson in the absence of interaction with the radiation field. When the radiation is emitted in a collision with a nonspherical nucleus, we can use for $G(\mathbf{r},\mathbf{r}')$ and $\Phi_{\mathbf{p}_1}$ the functions for the problem of diffraction by a stationary ellipsoid:

$$G(\mathbf{r},\mathbf{r}')=\frac{\exp\{ip_2|\mathbf{r}-\mathbf{r}'|\}}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

$$-\frac{p_2}{2\pi i}\int\frac{\exp\{ip_2|\mathbf{r}-\mathbf{s}|\}}{|\mathbf{r}-\mathbf{s}|}\,\frac{\exp\{ip_2|\mathbf{s}-\mathbf{r}'|\}}{4\pi|\mathbf{s}-\mathbf{r}'|}\,ds; \qquad (3)$$

$$\Phi_{\mathbf{p}_1}=e^{i\mathbf{p}_1\mathbf{r}}-\frac{p_1}{2\pi i}\int\frac{e^{ip_1|\mathbf{r}-\rho_1|}}{|\mathbf{r}-\rho_1|}\,d\rho_1. \qquad (4)$$

The integration is taken over those parts of planes, taken to pass through the center of the nucleon and to be perpendicular to the vector $\mathbf{r}-\mathbf{r}'$ for Eq. (3) and to $\mathbf{p}_1$ for Eq. (4), which lie inside the projections of the nuclear ellipsoid onto these planes. These regions of integration are functions of ϑ and ϕ — of the direction of the axis of symmetry of the nuclear ellipsoid relative to the chosen coordinate system. The projection is an ellipse. If in the plane in question we take the Y axis along the projection of the axis of symmetry, the equation of the ellipse is

$$X^2/a^2+Y^2/b^2=1,$$

$$b^2=a^2\xi^2(\vartheta);\quad \xi^2(\vartheta)=z^2\sin^2\vartheta+\cos^2\vartheta;\quad z=c/a, \qquad (5)$$

where c is half the length of the axis of symmetry and a is the radius of the largest circular section.

The use of the functions (3) and (4) is permissible in the adiabatic approximation, in which the energies of the particles are much larger than the rotational energy of the nucleus; this condition is fulfilled here, since the entire theory is valid only for E_1, $E_2 \gg \mu$.

When the meson is outside the range of the nuclear forces, the wave function of the whole system of π meson + nucleus, for the emission of a γ quantum, can be written in the adiabatic approximation in the form

$$\Phi_{\mathbf{p}_2}\Omega_0(\vartheta,\phi)=\int G(\mathbf{r},\mathbf{r}')\frac{2e}{i}\sqrt{\frac{2\pi}{\omega}}(\mathbf{j}\nabla\Phi_{\mathbf{p}_1})e^{-i\omega\mathbf{r}'}d\mathbf{r}'\,\Omega_0(\vartheta,\phi), \qquad (6)$$

where $\Omega_0(\vartheta,\phi)$ corresponds to the ground state of the nucleus. Starting with this formula, we can obtain the probability of emission of radiation in scattering with excitation of a definite rotational state, the total probability of radiation with scattering, and the probability of the "stopping" radiation.

Finding the asymptotic form of Eq. (6) for $r\to\infty$, we determine the amplitude for scattering with emission of radiation and with excitation of the n-th rotational state:

$$\langle\Omega_n(\vartheta,\phi)|M(\vartheta,\phi)|\Omega_0(\vartheta,\phi)\rangle$$

$$=\int\Omega_n(\vartheta,\phi)\frac{2e}{i}\sqrt{\frac{2\pi}{\omega}}\Phi^{*}_{\mathbf{p}_2}(\mathbf{j},\nabla\Phi_{\mathbf{p}_1})e^{-i\omega\mathbf{r}'}d\mathbf{r}'$$

$$\Omega_0(\vartheta,\phi)\sin\vartheta\,d\vartheta\,d\phi, \qquad (7)$$

$$\Phi_{\mathbf{p}_2}=e^{i\mathbf{p}_2\mathbf{r}}+\frac{p_2}{2\pi i}\int\frac{\exp\{-ip_2|\mathbf{r}-\rho_2|\}}{|\mathbf{r}-\rho_2|}\,d\rho_2,\quad \mathbf{p}_2=p_2\frac{\mathbf{r}}{r}.$$

The integration with respect to ρ_2 is taken over the projection of the nucleus onto a plane perpendicular to $\mathbf{p}_2$. The calculation of the probability of emission of a γ quantum in diffraction scattering with the excitation of rotational states is made by means of Eq. (7), in analogy with the calculation of the probability of radiation in scattering made by Landau and Pomeranchuk in Ref. 1. On carrying out these calculations and using the relations

$$E_1 \sim p_1; \quad E_2 \sim p_2; \quad E_1,\ E_2,\ \omega \gg \mu;$$

$$\mathbf{p}_2 = (\mathbf{p}_2\mathbf{p}_1)\,\mathbf{p}_1 / p_1^2 + p_2\theta'; \quad \omega = (\omega\mathbf{p}_1)\,\mathbf{p}_1 / p_1^2 + \omega\theta\,,$$

$$\theta \text{ and } \theta' \perp \mathbf{p}_1; \ |\theta|,\ |\theta'| \lesssim \mu / E,$$

we get, neglecting terms of the order μ/E in comparison with unity:

$$d\sigma_{n0} = \frac{e^2}{\pi^2}\,\frac{d\omega\, do_\omega}{\omega}\,\frac{E_2\, do_{\mathbf{p}_2}}{E_1}\,a^2 |F|\,\frac{|J_{n0}(|\omega\theta + E^2\theta'|\,a;\,z)|^2}{|\omega\theta + E_2\theta'|^2}$$
$$\times \left|\frac{(\theta-\theta')\,E_1}{(\mu / E_2)^2 + (\theta-\theta')^2} - \frac{\theta E_2}{(\mu / E_1)^2 + \theta^2}\right|^2, \tag{8}$$

F is the form factor of the π meson,[1] and

$$J_{n0}(ka;\,z) = \int \Omega_n(\vartheta,\ \Phi)\,\xi^2(\vartheta)\,\frac{J_1\left(ka\sqrt{\xi^2(\vartheta)\sin^2\Phi + \cos^2\Phi}\right)}{\sqrt{\xi^2(\vartheta)\sin^2\Phi + \cos^2\Phi}}\,\Omega_0(\vartheta,\Phi)\sin\vartheta\, d\vartheta\, d\Phi. \tag{9}$$

As has been shown by Drozdov,[4b] the functions (9) determine the excitation of the rotational states of nuclei by fast particles. He has computed them by numerical integration for a number of states of a nucleus with spin zero [$\Omega(\vartheta,\phi)$ are then ordinary spherical harmonics] and for various values of the parameters ka and z. Their main property is a rapid decrease with increase of the number n.

Drozdov's results[4] are entirely applicable in the present case also. The only difference is that now k is not the transverse momentum $p_2\theta'$ of the scattered particle, but is given by $|\omega\theta + E_2\theta'|$, which is equal to the transverse momentum only for small ω.

For such high energies it is of more interest to consider the total cross-section for emission of radiation in collisions with the nonspherical nucleus

$$d\sigma = \sum_n d\sigma_{n0}. \tag{10}$$

These cross-sections have the same form for all axially symmetrical nuclei with a given ratio of the semiaxes, independent of the spin,[5] i.e., independent of the lowest rotational state:

$$d\sigma = \frac{e^2}{\pi^2}\,\frac{d\omega do_\omega}{\omega}\,\frac{E_2 do_{\mathbf{p}_2}}{E_1}\,a^2 |F|^2\,\frac{I_{00}(|\omega\theta + E_2\theta'|\,a;\,z)}{|\omega\theta + E_2\theta'|^2}$$
$$\times \left|\frac{(\theta-\theta')\,E_1}{(\mu^2 / E_2^2) + (\theta-\theta')^2} - \frac{\theta E_2}{(\mu^2 / E_1^2) + \theta^2}\right|^2,$$

$$I_{00}(ka;\,z) \tag{11}$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\Phi \int_{-1}^{+1} dx\,\xi^2(x)\left|\frac{J_1\left(ka\sqrt{\xi^2(x)\sin^2\Phi + \cos^2\Phi}\right)}{\sqrt{\xi^2(x)\sin^2\Phi + \cos^2\Phi}}\right|^2.$$

The function $I_{00}(ka;\,z)$ determines the total cross-section for scattering of the particle by the nonspherical nucleus with excitation of all rotational states.[4] It has also been calculated by Drozdov[4b] by numerical integration for a number of values of the parameters ka and z.

For $z = 1$ the formulas of (11) give the result obtained in Ref. 1 for the spherical nucleus.

Introducing the dimensionless quantities s and η,

$$\theta = (\mu / E_1)\,s,\ \ \theta' - \theta = (\mu / E_2)\,\eta;\ \ s, \eta \lesssim 1,$$

we get instead of (11)

$$d\sigma = \frac{e^2}{\pi^2}\,\frac{d\omega}{\omega}\,\frac{E_2}{E_1}\,a^2 |F|^2\,\frac{I_{00}(\mu a\,|s+\eta|;\,z)}{|s+\eta|^2}$$
$$\times \left|\frac{s}{1+s^2} + \frac{\eta}{1+\eta^2}\right|^2 ds\,d\eta\,. \tag{12}$$

For heavy nuclei ($\mu a \gg 1$) and large angles s and $\eta \sim 1$, Eq. (12) is considerably simplified if we replace the Bessel function J_1 by the asymptotic expression for large arugments:

$$d\sigma = \frac{e^2}{\pi^3}\,\frac{d\omega}{\omega}\,\frac{E_2}{E_1}\,\frac{a}{\mu}\,\frac{|F|^2}{|s+\eta|^2}\left|\frac{s}{1+s^2} + \frac{\eta}{1+\eta^2}\right|^2 ds\,d\eta\, I; \tag{13}$$

$$I = \begin{cases} \dfrac{1}{4}\left\{1 + \dfrac{z^2+2}{\sqrt{z^2-1}}\ \sin^{-1}\ \dfrac{\sqrt{z^2-1}}{z}\right\} & z \geqslant 1 \\[2ex] \dfrac{1}{4}\left\{1 + \dfrac{z^2+2}{\sqrt{1-z^2}}\sinh^{-1}\dfrac{\sqrt{1-z^2}}{z}\right\} & z \leqslant 1. \end{cases} \tag{14}$$

Equation (13) can be written in the form

$$d\sigma = d\sigma_{\mathrm{sph}} I, \tag{15}$$

where $d\sigma_{\mathrm{sph}}$ is the cross-section for emission of radiation in scattering by a spherical nucleus of radius a [Ref. 1, Eq. (30)]. As is shown in Ref. 1, the main contribution to the total cross-section comes from values of s and η of the order of unity. Thus the total cross-section for heavy nuclei can be obtained with $F = 1$ in Eq. (13) [Ref. 1, Eq. (32)]:

$$\sigma = \sigma_{\mathrm{sph}} I \approx 2{,}3\, e^2\,\frac{E_2}{E_1}\,\frac{a}{\omega\mu}\, I. \tag{16}$$

Consequently, for heavy nuclei and large angles the cross-section for bremsstrahlung in scattering by nonspherical nuclei differs from the corresponding cross-section for the spherical nucleus only by the factor (14). In the general case the angular distribution for the nonspherical nucleus differs from that for the spherical nucleus by the fact that for $z \neq 1$ the function $I_{00}(ka; z)$, and thus also the cross-section, does not become zero for any value of $k \neq 0$. For spherical nuclei, on the other hand, with $z = 1$, $I_{00}(ka; 1) = J_1^2(ka)$, and thus the corresponding cross-section becomes zero for certain values of k. Curves showing the behavior of the function I_{00} for various values of z and the angles are given in Ref. 4b.

The "stopping" radiation from a nonspherical nucleus can be calculated if one uses Eq. (6) to find the flux of π mesons incident on the nucleus, integrates this flux over the surface of the nucleus as is done in Ref. 1, and then averages over the ground state of rotation of the nucleus. The cross-section obtained does not depend on the spin of the ground state of the nucleus, and differs only by a factor (a function of z) from the corresponding cross-section for the spherical nucleus of radius $R = a$:

$$d\sigma = d\sigma_{c\phi} I_1 = \frac{e^2}{\pi} a^2 \frac{\theta^2 do_\omega d\omega}{[(\mu / E_1)^2 + \theta^2]^2} \frac{E_1 - \omega}{E_1 \omega} |F|^2 I_1; \tag{17}$$

$$I_1 = \begin{cases} \frac{1}{2}\left\{1 + \frac{z^2}{\sqrt{z^2 - 1}} \sin^{-1} \frac{\sqrt{z^2 - 1}}{z}\right\} & z \geqslant 1 \\ \frac{1}{2}\left\{1 + \frac{z^2}{\sqrt{1 - z^2}} \sinh^{-1} \frac{\sqrt{1 - z^2}}{z}\right\} & z \leqslant 1, \end{cases} \tag{18}$$

$\pi a^2 I_1$ is the average area of the projection of the nuclear ellipsoid onto the plane perpendicular to the direction of motion of the incident π mesons.

2. PRODUCTION OF PAIRS OF PI MESONS BY AN INCIDENT GAMMA QUANTUM

This process can also be dealt with by means of Eq. (1), as a process of "absorption" of the γ quantum by a π meson with energy and momentum $-E_1$, $-\mathbf{p}_1$, where E_1 and $\mathbf{p}_1$ are the energy and momentum of one member of the pair.*

When the meson is outside the nucleus, the wave function of the system π meson + nucleus, after the "absorption" of the γ quantum, is

$$\frac{2e}{i} \sqrt{\frac{2\pi}{\omega}} \int G(\mathbf{r}, \mathbf{r}') (\mathbf{j}, \nabla \Phi_{-\mathbf{p}_1}(\mathbf{r}') e^{i\omega \mathbf{r}'} d\mathbf{r}' \Omega_0(\vartheta, \phi), \tag{19}$$

$G(\mathbf{r}, \mathbf{r}')$ is given by Eq. (3), and

$$\Phi_{-\mathbf{p}_1}(\mathbf{r}) = e^{-i\mathbf{p}_1\mathbf{r}} - \frac{p_1}{2\pi i} \int \frac{\exp\{ip_1 |\mathbf{r} - \mathbf{s}|}{|\mathbf{r} - \mathbf{s}|} d\mathbf{s}.$$

The integration is taken over the projection of the nucleus onto the plane perpendicular to $\mathbf{p}_1$ [Eq. (5)].

The calculations are carried out in complete analogy with the case of the spherical nucleus,[2a, b] with the nonspherical shape taken into account by a formula analogous to Eq. (7). The calculation can be carried to completion only for the case of heavy nuclei ($\mu a \gg 1$, $\mu c \gg 1$). We proceed at once to give the results. Using the notation of Pomeranchuk[2] we have

$$\mathbf{p}_i = p_i \frac{\boldsymbol{\omega}}{\omega}\left(1 - \frac{k_i^2}{2p_i}\right) + \mathbf{k}_i, \quad \boldsymbol{\omega}\mathbf{k}_i = 0, \ k_i \ll p_i,$$

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}, \quad i = 1, 2.$$

The cross-section for pair production with excitation of the n-th rotational state is

$$d\sigma_{n0}(\mathbf{p}_1, \mathbf{p}_2) = \frac{e^2}{8\pi^2} a^2 \frac{E_1 E_2}{\omega^3} \frac{|F_{+-}|^2}{k^2} \left| \frac{\mathbf{k}_2 - \mathbf{k}_1}{\mu^2 + (\mathbf{k}_2 - \mathbf{k}_1)^2/4} + \frac{2\mathbf{k}_1}{\mu^2 + k_1^2} - \frac{2\mathbf{k}_2}{\mu^2 + k_2^2} \right|^2 |J_{n0}(ka; z)|^2 \, dE_1 \, d\mathbf{k}_1 \, d\mathbf{k}_2. \tag{20}$$

The function $J_{n0}(ka; z)$ is defined by Eq. (9). With regard to the form factor F_{+-}, see Ref. 2. The cross-section for pair production from a nonspherical nucleus with excitation of any rotational state is

$$d\sigma(\mathbf{p}_1, \mathbf{p}_2) = \frac{e^2}{8\pi^2} a^2 \frac{E_1 E_2}{\omega^3} \frac{|F_{+-}|^2}{k^2} \left| \frac{\mathbf{k}_2 - \mathbf{k}_1}{\mu^2 + (\mathbf{k}_2 - \mathbf{k}_1)^2 / 4} + \frac{2\mathbf{k}_1}{\mu^2 + k_1^2} - \frac{2\mathbf{k}_2}{\mu^2 + k_2^2} \right|^2 \times I_{00}(ka; z) \, dE_1 \, d\mathbf{k}_1 \, d\mathbf{k}_2. \tag{21}$$

The function I_{00} is defined in Eq. (11).

Thus the changes in the angular distributions for pair production from a nonspherical nucleus, as compared with the case of the spherical nucleus, are the same as for the bremsstrahlung.

The total cross-sections can be calculated, as in Ref. 2b, beginning with Eq. (21), if we set $|F_{+-}|^2$ equal to 1. These cross-sections differ from those calculated by Pomeranchuk by the geometrical factor (18) and the replacement of R^2 by a^2. For example, integrating the pair-production cross-section (21) over all $\mathbf{k}_1$ and $\mathbf{k}_2$, up to certain maximum values, and over all possible energies of the pair, we get:

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*It is well known that the matrix element for pair production is obtained from that for bremsstrahlung by the replacement $\omega \to -\omega$, $\boldsymbol{\omega} \to -\boldsymbol{\omega}$ and change of sign of the energy and momentum of the incident particle.[6]

$$\sigma = \frac{e^2a^2}{12}\left[\ln\frac{\mu^2+b_{\max}^2}{\mu^2}+\frac{\mu^2}{\mu^2+b_{\max}^2}\right]I_1,$$

$$b_{\max} = \frac{1}{2}\,|\,\mathbf{k}_1-\mathbf{k}_2\,|_{\max}. \qquad (22)$$

3. PRODUCTION OF A PAIR OF PI MESONS BY A GAMMA QUANTUM WITH SUBSEQUENT ABSORPTION OF ONE MEMBER OF THE PAIR BY A NONSPHERICAL NUCLEUS

This process can also be treated by means of Eq. (19), finding the flux falling on unit area of the nucleus, integrating over the surface of the nucleus and averaging over the rotational state and over the polarization of the γ quantum. Thus we get, in analogy with Ref. 3b:

$$d\sigma = \frac{e^2}{2\pi^2}\,\frac{E_1(\omega-E_1)}{\omega^3}\,d\mathbf{k}_1\,dE_1\int|\,\Omega_0(\vartheta,\ \Phi)\,|^2\sin\vartheta\,d\vartheta\,d\Phi$$

$$\times\int d\rho\left|\frac{\mathbf{k}_1}{\mu^2+k_1^2}\right.$$

$$\left.-\frac{1}{4\pi^2}\int ds\,d\mathbf{g}\,\frac{\mathbf{g}}{\mu^2+g^2}\exp\{i(\mathbf{g}-\mathbf{k}_1,\mathbf{s}-\boldsymbol{\rho})\}\right|^2|\,F_{+-}\,|^2. \quad (23)$$

We integrate with respect to ds and dρ over the projection of the nucleus onto the plane perpendicular to $\mathbf{p}_1$, and with respect to g over the whole of the plane perpendicular to $\boldsymbol{\omega}$:

$$\frac{\mathbf{k}_1}{\mu^2+k_1^2}-\frac{1}{4\pi^2}\int ds\,d\mathbf{g}\,\frac{\mathbf{g}}{\mu^2+g^2}\exp\{i(\mathbf{g}-\mathbf{k}_1,\mathbf{s}-\boldsymbol{\rho})\} =$$

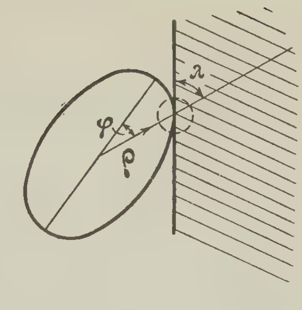
$$=\frac{\mu}{2\pi}\,\nabla_{\mathbf{k}_1}\int\limits_{s\geqslant R(\vartheta,\varphi)} ds\exp\{-i\mathbf{k}_1(\mathbf{s}-\boldsymbol{\rho})\}\,\frac{K_1(\mu\,|\,\mathbf{s}-\boldsymbol{\rho}\,|)}{|\,\mathbf{s}-\boldsymbol{\rho}\,|}. \qquad (24)$$

$K_1(x)$ is a solution of Bessel's equation for $n = 1$ which falls off exponentially at infinity. The integral (24) is taken over the part of the plane perpendicular to $\mathbf{p}_1$ that is outside the projection of the nuclear ellipsoid; ϑ is the angle between the axis in this plane, measured from the axis of symmetry (the diagram shows the projection of the nuclear ellipsoid onto the plane perpendicular to $\mathbf{p}_1$).

From Eq. (5) we have:

$$R(\vartheta,\varphi) = a\xi(\vartheta)\,/\sqrt{\xi^2(\vartheta)\cos^2\varphi+\sin^2\varphi}. \qquad (25)$$

The integral in Eq. (24) can be calculated for heavy nuclei ($\mu a \gg 1$, $\mu c \gg 1$). Since the important region for this integral is $|\,\mathbf{s}-\boldsymbol{\rho}\,| \sim 1/\mu$, neglecting terms of order $1/\mu a$ we can use an integration (for each vector $\boldsymbol{\rho}$ making an angle φ with the projection of the axis of symmetry) taken not over the part of the plane outside the ellipse at the point of intersection with the extension of the vector $\boldsymbol{\rho}$ (see diagram); this makes it possible to do the integration. We then integrate the square of the absolute value of the resulting expression with respect to $\boldsymbol{\rho}$ over the interior of the ellipse (25),



$$\int d\boldsymbol{\rho}\left|\frac{\mu}{2\pi}\,\nabla_{\mathbf{k}_1}\int ds\exp\{-i\mathbf{k}_1(\mathbf{s}-\boldsymbol{\rho})\}\,\frac{K_1(\mu\,|\,\mathbf{s}-\boldsymbol{\rho}\,|)}{|\,\mathbf{s}-\boldsymbol{\rho}\,|}\right|^2$$

$$=\frac{1}{8}\,\frac{1}{\mu^2+k_1^2}\int\limits_{-\pi}^{+\pi}d\varphi\left\{1\right.$$

$$\left.+\frac{k_1^2\cos^2(\lambda+\varphi+\Phi)}{\mu^2+k_1^2\cos^2(\lambda+\varphi+\Phi)}\right\}\frac{R(\vartheta,\varphi)}{\sin\lambda\,[\mu^2+k_1^2\cos^2(\lambda+\varphi+\Phi)]^{1/2}}, \qquad (26)$$

where λ is the angle between the direction of $\boldsymbol{\rho}$ and the tangent to the ellipse (see drawing):

$$\sin\lambda = [\xi^2(\vartheta)\cos^2\varphi+\sin^2\varphi]\,/\,[\sin^2\varphi+\cos^2\varphi\,\xi^4(\vartheta)]^{1/2}.$$

Just as was found in Eq. (11), averaging of the expression (26) over the lowest rotational state gives a result that is independent of the spin of the ground state,[5] and the averaging reduces simply to averaging over ϑ and ϕ.

The result differs from that for the case of the spherical nucleus[3a] only by a factor that is a function of z:

$$d\sigma = \frac{e^2}{4\pi^2}\,\frac{E_1(\omega-E_1)}{\omega^3}\,d\eta\,dE_1\,\frac{a}{\mu}\,\frac{|\,F_{+-}\,|^2}{(1+\eta^2)^{3/2}}\{2K(\varepsilon)-E(\varepsilon)\}\,I_2,$$

$$\eta = k_1/\mu;\ \ \varepsilon = \eta/\sqrt{1+\eta^2} \qquad (27)$$

(the notation is that of Vdovin[3a]); K and E are the complete elliptic integrals of the first and second kinds; and

$$I_2 = \frac{1}{4\pi}\int\limits_{-1}^{+1}dx\int\limits_{-\pi}^{+\pi}d\varphi\,\frac{\xi(x)\,[\sin^2\varphi+\cos^2\varphi\,\xi^4(x)]^{1/2}}{[\sin^2\varphi+\cos^2\varphi\,\xi^2(x)]^{3/2}}; \qquad (28)$$

$$I_2 = \begin{cases} {}^1\!/_2\,\{z+(z^2-1)^{-1/2}\sinh^{-1}(z^2-1)^{1/2}\} & z\geqslant 1\\ {}^1\!/_2\,\{z+(1-z^2)^{-1/2}\cdot\sin^{-1}(1-z^2)^{1/2}\} & z\leqslant 1. \end{cases}$$

Thus the angular distribution and the total cross-section for the nonspherical nucleus differ from those for the spherical nucleus[3b] only by the factor (28).

Let us now consider the conditions for the applicability of the formulas that have been obtained. The integration of (24) over a half-plane instead of over the part of the plane outside the ellipse amounts to the neglect of terms of order $1/\mu a \ll 1$ only if $\rho_\varphi \gg 1/\mu$, where ρ_φ is the radius of curvature of the ellipse at the point in question. For the spherical nucleus this condition is just $\mu R \gg 1$. For the ellipse the curvature is variable, and the minimum radius of curvature of the projection of the nuclear ellipsoid is $z^2 a$ for $z < 1$ and a/z for $z > 1$. It would appear that the treatment that has been carried out is valid only for

$$z^2 \gg 1/\mu a, \qquad z < 1$$
$$1/z \gg 1/\mu a, \qquad z > 1. \tag{29}$$

But in actual fact Eqs. (27) and (28) are valid over a range of values of z considerably larger than that given by the conditions (29), since the radius of curvature ρ_φ fails to be much larger than $1/\mu$ only in small ranges of angles given by $\Delta\varphi \sim z^{-2} \times (z/\mu a)^{1/3}$ for $z > 1$ and $\Delta\varphi \sim z(z/\mu a)^{1/3}$ for $z < 1$, and the contribution from these intervals to the integral (26), for $z \gg 1$ and $z \ll 1$, is small in comparison with the whole integral (26).

Thus (27) and (28) can be used for heavy nuclei with $\mu a \gg 1$, $\mu c \gg 1$, independently of the value of z.

If both mesons of the pair are absorbed by the heavy nucleus, producing a "star," the cross-section for this process with a nonspherical nucleus differs from the corresponding cross-section with a spherical nucleus[3c] by the factor (18) and the replacement of R by a.

In conclusion the writer expresses his deep gratitude to I. M. Shmushkevich for suggesting the problem and for valuable discussions.

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POSSIBLE EXPERIMENTS ON INELASTIC SCATTERING OF NUCLEONS. I

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Possible experiments are considered for the investigation of inelastic collisions of nucleons with the reaction scheme $N + N \rightarrow \pi + N + N'$, which represents the most intense inelastic processes in the ~ 650-Mev region. Some nontrivial experiments are suggested on the angular correlation of π mesons and nucleons based on the existence of azimuthal asymmetry in the emission of these particles. Analogous effects in a polarized nucleon beam are also considered. The calculations are performed in terms of ten inelastic transition amplitudes which are assumed to provide the principal contribution to the total cross section for $N + N \rightarrow \pi + N + N'$ at 650 Mev.

NUCLEON scattering experiments ~ 650 Mev which are intended to serve as the basis of a quantitative phase analysis include both elastic and inelastic scattering. We are already well acquainted with the program of experiments on elastic scattering including the observation of various polarization effects.[1,2] However, for inelastic processes whose total cross section at ~ 650 Mev is about equal to the total elastic cross section[3] the experimental program has not yet been completely considered. Theoretical articles have been concerned with only the reaction $p + p \rightarrow \pi^+ + d$,[4-8] and have not considered the more intense inelastic processes in the ~ 650-Mev region: $p + p \rightarrow \pi^+ + n + p$ and $p + p \rightarrow \pi^0 + p + p$. It is the purpose of the present article to fill this gap and to determine the experimental possibilities for the observation of the reactions represented by $N + N \rightarrow \pi + N + N'$ In the first part the angular π-meson distribution and the π-meson-nucleon angular correlation are analyzed for both polarized and unpolarized nucleon beams. Calculations are carried through for the two reactions $p + p \rightarrow \pi^+ + n + p$ and $p + p \rightarrow \pi^0 + p + p$.

1. GENERAL COMMENTS

We know that strong absorption by a scattering center complicates substantially the character of elastic scattering. When it is necessary to determine the phase shifts of different partial waves through observation of an elastic process, the absorption intensities of these partial waves are also required. The latter information can be obtained by analyzing the angular and spin characteristics of secondary particle emission for all possible elastic process channels. The most complicated case is that of π-meson production: $N + N \rightarrow \pi + N + N'$, which is characterized by the final emission of three particles.

Any specific instance of this process for which the energy of the incident nucleon is known can be characterized completely by giving the energy of one of the particles, the polar angles (θ, φ) of the direction of flight of this particle in the center-of-mass system of the colliding nucleons, and the polar angles of emission for the remaining two particles in their center-of-mass system. The differential cross section will thus depend on five arguments.

For the purpose of determining the angular part of the wave function for the three ultimate particles, the three-body problem is usually reduced to a two-body problem by introducing two subsystems. This can be accomplished in two ways. In the first, two nucleons constitute one subsystem. This division enables us to add separately the nucleon spins in one subsystem and the orbital angular momenta in each subsystem. Such a procedure involving two subsystems is therefore usually called an ℓs-scheme. The angular part of the combination of two subsystems is written as

$$\Psi^{(ls)} = \sum_{m+\mu=M} (l_\pi j m \mu \,|\, l_\pi j J M)\, Y_{l_\pi}^{m}(\pi)\, Y_{j}^{\mu}(1,2).$$

Considering the orbital angular momentum of relative motion of the two nucleons and their spins, we finally obtain

$$\Psi^{(ls)} = \sum_{\substack{m+\mu=M \\ m_l+m_s=\mu \\ m_{s_1}+m_{s_2}=m_s}}\sum\sum (l_\pi jm\mu \mid l_\pi jJM)(lsm_l m_s \mid lsj\mu)$$
$$\times (s_1 s_2 m_{s_1} m_{s_2} \mid s_1 s_2 s m_s) \times Y_{l_\pi}^{m}(\pi)\, Y_l^{m_l}(1, 2)\, \chi_{s_1}^{m_{s_1}}(1)\, \chi_{s_2}^{m_{s_2}}(2), \tag{1}$$

where

$$Y_{l_\pi}^{m}(\pi) \equiv Y_{l_\pi}^{m}(\theta_\pi, \varphi_\pi), \quad Y_l^{m_l}(1,2) \equiv Y_l^{m_l}(\theta_{12}, \varphi_{12})$$

are the spherical harmonics of the respective arguments. The angles θ_π, φ_π characterize the outgoing direction of the π meson in the c.m.s. of the two colliding nucleons and the angles θ_{12}, φ_{12} denote the outgoing direction of nucleons "1" in the coordinate system which is associated with the center of mass of inelastically scattered nucleons. Nucleon "2" is ejected in the opposite direction so that $\theta_{21} = \pi - \theta_{12}$ and $\varphi_{21} = \varphi_{12} + \pi$.

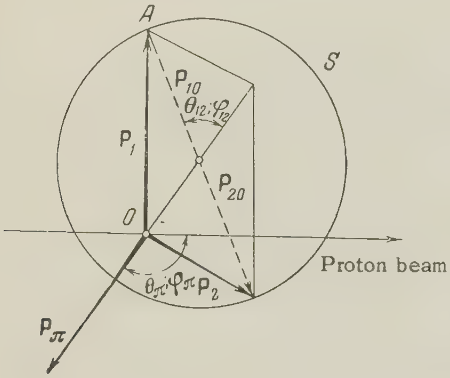

FIG. 1

The representation of the angular part of the wave function by Eq. (1) is most suitable for experiments which observe the π mesons. On the other hand, when we are interested in experiments where only the nucleons are observed, it is most convenient to employ the jj-scheme of two subsystems where one of the subsystems contains the π meson and one of the nucleons. It must be emphasized that the ℓs- and jj-schemes for writing the angular part correspond to the two possible expansions of the angular part of the three-particle wave function in the eigenfunctions of the two subsystems. Which of these two expansions is more suitable, i.e., which contains the smaller number of amplitudes that provide the principal contribution to the cross section, depends on the mechanism of π-meson production and on the character of interaction of the ejected particles, and can only be decided experimentally.

The angular part of the three-particle wave function in the jj-scheme is written as follows:

$$\Psi^{(jj)} = \sum_{\substack{m+m_j=M \\ m_l+m_{s_1}=m_j \\ m_2+m_{s_2}=m}}\sum\sum (jLm_j m \mid jLJM)(l_2 s_2 m_2 m_{2s} \mid l_2 s_2 Lm)$$
$$\times (l_\pi s_1 m_l m_{1s} \mid l_\pi s_1 j m_j) \times Y_{l_2}^{m_2}(2)\, Y_{l_\pi}^{m_l}(\pi 1)\, \chi_{s_1}^{m_{s_1}}(1)\, \chi_{s_2}^{m_{s_2}}(2). \tag{2}$$

Figure 1 is a vector diagram of the secondary particle momenta for the reaction N + N → π + N + N′ in the c.m.s. of the colliding nucleons. When the secondary nucleons can be considered approximately, neglecting their relativistic mass increase, the geometrical locus A of the end of a nucleon vector is a sphere of radius $R = [M(W - \sqrt{p^2+m^2} - p^2/4]^{1/2}$, where W is the total kinetic energy in the c.m.s., p is the momentum of the π meson, and M and m are the nucleon and π-meson masses. For this case it is easily shown that the differential of the phase volume density for two particles in the ℓs-scheme, $d^2N/d\Omega(\theta_{12}, \varphi_{12})dW$, is independent of the angles θ_{12}, φ_{12} and is determined by the magnitude of the radius R. For the jj-scheme this no longer holds true: $d^2N/d\Omega(\theta_{\pi 1}, \varphi_{\pi 1})dW$ depends on $\theta_{\pi 1}$ but is independent of $\varphi_{\pi 1}$.

2. STATEMENT OF THE PROBLEM

In our calculations we have used Berestetskii's formalism[6] for the study of polarization effects in inelastic collisions of nucleons.

The initial function for two protons, one of which (in the target) is unpolarized, has the following form:

$$\Psi_{\text{init}} = \varphi_1 + \varphi_2,$$

$$\varphi_1 = \frac{\varepsilon_1}{\sqrt{2}}\sqrt{4\pi}\Big\{\sum_{l=0}^{\infty}(-i)^l\sqrt{2l+1}\,[1 - (-1)^l]\Big[q_1\sum_J^{\infty}(l101 \mid l1J1)\,Y_{J(l1)}^{1} + \frac{q_2}{\sqrt{2}}\sum_J^{\infty}(l100 \mid l1J0)\,Y_{J(l1)}^{0}\Big]$$
$$+ \sum_{l=0}^{\infty}(-i)^l\sqrt{2l+1}\,[1 + (-1)^l]\,\frac{q_2}{\sqrt{2}}(l000 \mid l0l0)\,Y_{l(l0)}^{0}\Big\}; \tag{3}$$

$$\varphi_2 = \frac{\varepsilon_2}{\sqrt{2}}\sqrt{4\pi}\Big\{\sum_{l=0}^{\infty}(-i)^l\sqrt{2l+1}\,[1 - (-1)^l] \times \Big[q_2\sum_J^{\infty}(l10-1 \mid l1J-1)\,Y_{J(l1)}^{-1} + \frac{q_1}{\sqrt{2}}\sum_J^{\infty}(l100 \mid l1J0)\,Y_{J(l1)}^{0}\Big] \tag{4}$$
$$- \sum_{l=0}^{\infty}(-i)^l\sqrt{2l+1}\,[1 + (-1)^l]\,\frac{q_1}{\sqrt{2}}(l000 \mid l0l0)\,Y_{l(l0)}^{0}\Big\}.$$

The coefficients ϵ_1 and ϵ_2 are subject to the condition $|\epsilon_1|^2 = |\epsilon_2|^2 = \frac{1}{2}$; $\epsilon_1^*\epsilon_2 = 0$ and describe the unpolarized target, while $q_1 = \cos\frac{\vartheta}{2}\, e^{-i\delta/2}$; $q_2 = \sin\frac{\vartheta}{2}\, e^{i\delta/2}$ are coefficients that describe the spin function of a proton beam which is completely polarized in the direction (ϑ, δ).

In this article, the possibilities of different independent experiments are considered only for one special case, which takes into account a limited number of inelastic transition amplitudes. Certain qualitative generalizations are then based on this special case. Specifically, we shall assume that for the ℓs-scheme with incident proton energies ~ 650 Mev it is sufficient to consider the transitions listed in the table.

The amplitude of each transition is a function of the π-meson momentum and incident proton energy. The expression for the cross sections will not contain the momenta of the nucleons, because the phase volume of a subsystem of two nucleons does not depend on their emission angles $(\theta_{12}, \varphi_{12})$. In other words, the final angular and momentum parts are completely separated.

3. DISCUSSION OF RESULTS

We have calculated the mean values of the spin tensors which are directly related to the experimentally observed physical quantities. For example, the mean value of the zero-rank spin tensor $\langle T_{00}\rangle$ determines the differential cross section for the emission of a π meson in the direction $(\theta_\pi, \varphi_\pi)$, when the nucleons are emitted in the direction $(\theta_{12}, \varphi_{12})$ measured from the π-meson direction in the c.m.s. of the two nucleons. The spin tensor $\langle T_{00}\rangle$ and the differential cross section are then related by

$$\langle T_{00}\rangle (\theta_\pi, \varphi_\pi;\, \theta_{12}, \varphi_{12};\, p_\pi) = \frac{k^2}{\rho_E}\, \frac{d^3\sigma(\theta_\pi, \varphi_\pi;\, \theta_{12}, \varphi_{12};\, p_\pi)}{d\Omega(\theta_\pi, \varphi_\pi)\, d\Omega(\theta_{12}, \varphi_{12})\, dp_\pi}. \qquad (5)$$

Here $k = 1/\lambda\!\!\!^-$, p_π is the momentum of the π meson in the c.m.s. of the colliding nucleons, ρ_E is the phase volume of the three particles, $\rho_E = p_\pi^2 R(p_\pi;\, W)$.

Type of transition	Transition scheme	Transition amplitude
σ_{10}	${}^1S_0 \to ({}^3S_1p)_0$	c_1
	${}^1D_2 \to ({}^3S_1p)_2$	c_2
	${}^3P_0 \to ({}^1P_1p)_0$	b_1
	${}^3P_1 \to ({}^1P_1p)_1$	b_2
	${}^3P_2 \to ({}^1P_1p)_2$	b_3
	${}^3F_2 \to ({}^1P_1p)_2$	b_4
σ_{11}	${}^3P_0 \to ({}^3P_1p)_0$	d_1
	$\nearrow ({}^3P_0p)_1$	d_2
	${}^3P_1 \to ({}^3P_1p)_1$	d_3
	$\searrow ({}^3P_2p)_2$	d_4

The final wave function is

$$F_1 = [A_1\chi_0^0 + \alpha_1\chi_1^1 + \beta_1\chi_1^0 + \gamma_1\chi_1^{-1}]/\sqrt{4\pi},$$
$$F_2 = [A_2\chi_0^0 + \alpha_2\chi_1^1 + \beta_2\chi_1^0 + \gamma_2\chi_1^{-1}]/\sqrt{4\pi}, \qquad (6)$$

where the coefficients A_1, α_1 etc. are related to the transition amplitudes as follows:

$$A_1 = -iq_1\frac{3}{\sqrt{2}}\Big[\Big(b_- + \frac{\sqrt{3}}{2}b_2\Big)\sin\theta_\pi\cos\theta_{12}e^{i\varphi_\pi} + \Big(b_- - \frac{\sqrt{3}}{2}b_2\Big)\cos\theta_\pi\sin\theta_{12}e^{i\varphi_{12}}\Big] - 3iq_2\Big[\Big(\frac{b_1}{\sqrt{6}} - b_+\Big)\sin\theta_\pi\sin\theta_{12}\cos(\varphi_\pi - \varphi_{12})$$
$$+ \Big(\frac{b_1}{\sqrt{6}} + 2b_+\Big)\cos\theta_\pi\cos\theta_{12}\Big];$$

$$\alpha_1 = iq_2\Big[\frac{c_-}{2i}\sin\theta_\pi e^{-i\varphi_\pi} + \frac{1}{2}\sqrt{\frac{3}{2}}d_1\big(\cos\theta_\pi\sin\theta_{12}e^{-i\varphi_{12}} - \sin\theta_\pi\cos\theta_{12}e^{-i\varphi_\pi}\big)\Big]$$
$$+ \frac{3}{2}\cdot iq_1\Big[\Big(\sqrt{\frac{3}{2}}d_3 - \frac{3}{\sqrt{10}}d_4\Big)\cos\theta_\pi\cos\theta_{12} - D_-\sin\theta_\pi\sin\theta_{12}e^{i(\varphi_\pi - \varphi_{12})} - \frac{3}{\sqrt{10}}d_4\sin\theta_\pi\sin\theta_{12}e^{-i(\varphi_\pi - \varphi_{12})}\Big];$$

$$\beta_1 = iq_2\Big[-\frac{c_+}{2i}\cos\theta_\pi + i\frac{\sqrt{3}}{2}d_1\sin\theta_\pi\sin\theta_{12}\sin(\varphi_\pi - \varphi_{12})\Big]$$
$$+ \frac{3}{2}iq_1\Big[\Big(d_2 - \frac{d_4}{\sqrt{5}}\Big)\sin\theta_\pi\cos\theta_{12}e^{i\varphi_\pi} + \Big(\frac{\sqrt{3}}{2}d_3 + \frac{3}{\sqrt{20}}d_4\Big)\cos\theta_\pi\sin\theta_{12}e^{i\varphi_{12}}\Big];$$

$$\gamma_1 = iq_2\Big[-\frac{c_-}{2i}\sin\theta_\pi e^{i\varphi_\pi} + \frac{1}{2}\sqrt{\frac{3}{2}}d_1\big(\cos\theta_\pi\sin\theta_{12}e^{i\varphi_{12}} - \sin\theta_\pi\cos\theta_{12}e^{i\varphi_\pi}\big)\Big] + iq_1\frac{3}{2}D_+\sin\theta_\pi\sin\theta_{12}e^{i(\varphi_\pi + \varphi_{12})};$$

$$A_2 = \frac{3}{\sqrt{2}}iq_2\Big[\Big(b_- + \frac{\sqrt{3}}{2}b_2\Big)\sin\theta_\pi\cos\theta_{12}e^{-i\varphi_\pi} + \Big(b_- - \frac{\sqrt{3}}{2}b_2\Big)\cos\theta_\pi\sin\theta_{12}e^{-i\varphi_{12}}\Big]$$
$$- 3iq_1\Big[\Big(\frac{b_1}{\sqrt{6}} - b_+\Big)\sin\theta_\pi\sin\theta_{12}\cos(\varphi_\pi - \varphi_{12}) + \Big(\frac{b_1}{\sqrt{6}} + 2b_+\Big)\cos\theta_\pi\cos\theta_{12}\Big];$$

$$\alpha_2 = iq_1\left[-\frac{c_-}{2i}\sin\theta_\pi e^{-i\varphi_\pi} + \frac{1}{2}\sqrt{\frac{3}{2}}\,d_1\left(\cos\theta_\pi\sin\theta_{12}e^{-i\varphi_{12}} - \sin\theta_\pi\cos\theta_{12}e^{-i\varphi_\pi}\right)\right] - \frac{3}{2}\,iq_2 D_+\sin\theta_\pi\sin\theta_{12}e^{-i(\varphi_\pi+\varphi_{12})};$$

$$\beta_2 = iq_1\left[\frac{c_+}{2i}\cos\theta_\pi + i\frac{\sqrt{3}}{2}d_1\sin\theta_\pi\sin\theta_{12}\sin(\varphi_\pi-\varphi_{12})\right] + \frac{3}{2}iq_2\left[\left(d_2 - \frac{d_4}{\sqrt{5}}\right)\sin\theta_\pi\cos\theta_{12}\,e^{-i\varphi_\pi}\right.$$
$$\left. + \left(\frac{\sqrt{3}}{2}d_3 + \frac{3}{\sqrt{20}}\,d_4\right)\cos\theta_\pi\sin\theta_{12}e^{-i\varphi_{12}}\right];$$

$$\gamma_2 = iq_1\left[\frac{c_-}{2i}\sin\theta_\pi e^{i\varphi_\pi} + \frac{1}{2}\sqrt{\frac{3}{2}}\,d_1\left(\cos\theta_\pi\sin\theta_{12}e^{i\varphi_{12}} - \sin\theta_\pi\cos\theta_{12}e^{i\varphi_\pi}\right)\right] - \frac{3}{2}\,iq_2\left[\left(\sqrt{\frac{3}{2}}\,d_3 - \frac{3}{\sqrt{10}}\,d_4\right)\cos\theta_\pi\cos\theta_{12}\right.$$
$$\left. - D_-\sin\theta_\pi\sin\theta_{12}e^{-i(\varphi_\pi-\varphi_{12})} - \frac{3}{\sqrt{10}}\,d_4\sin\theta_\pi\sin\theta_{12}e^{i(\varphi_\pi-\varphi_{12})}\right]. \qquad (7)$$

Here we have the relations

$$c_+ = c_1 + \sqrt{10}c_2; \quad b_+ = \frac{b_3}{\sqrt{6}} + \frac{1}{2}b_4; \quad D_\pm = \frac{d_2}{\sqrt{2}} \pm \frac{1}{2}\sqrt{\frac{3}{2}}\,d_3 + \frac{d_4}{2\sqrt{10}};$$
$$c_- = c_1 - \sqrt{\frac{5}{2}}\,c_2; \quad b_- = -\frac{\sqrt{3}}{2}\,b_3 + \frac{1}{2}\,b_4. \qquad (8)$$

Furthermore,

$$\langle T_{00}\rangle = |F_1|^2 + |F_2|^2 = \frac{1}{4\pi}\{|A_1|^2 + |\alpha_1|^2 + |\beta_1|^2 + |\gamma_1|^2 + |A_2|^2 + |\alpha_2|^2 + |\beta_2|^2 + |\gamma_2|^2\}. \qquad (9)$$

The results can be represented by

$$\langle T_{00}\rangle = \langle T_{00}\rangle_{\text{unpol}} + P\langle T_{00}\rangle_{\text{pol}}. \qquad (10)$$

Here P is the degree of polarization of the incident proton beam along the y axis, and

$$\langle T_{00}\rangle_{\text{unpol}} = \frac{1}{4\pi}\,[f_0 + f_1\cos^2\theta_\pi + f_2\cos^2\theta_{12} + f_3\cos^2\theta_\pi\cos^2\theta_{12} + f_4\sin\theta_\pi\cos\theta_\pi\sin\theta_{12}\cos\theta_{12}\cos(\varphi_\pi-\varphi_{12})$$
$$+ f_5\sin^2\theta_\pi\sin^2\theta_{12}\cos 2(\varphi_\pi-\varphi_{12})]; \qquad (11)$$

$$\langle T_{00}\rangle_{\text{pol}} = \frac{1}{4\pi}\{\sin\theta_\pi\cos\theta_\pi\,[(g_0 + g_1\cos\theta_{12} + g_2\cos^2\theta_{12})\cos\varphi_\pi + g_3\sin^2\theta_{12}\cos(\varphi_\pi - 2\varphi_{12})]$$
$$+ \sin\theta_{12}\cos\theta_{12}\,[(g_4 + g_5\cos^2\theta_\pi)\cos\varphi_{12} + g_6\sin^2\theta_\pi\cos(\varphi_{12} - 2\varphi_\pi)] + \sin\theta_{12}\,[(g_7 + g_8\cos^2\theta_\pi)\cos\varphi_{12}$$
$$+ g_9\sin^2\theta_\pi\cos(\varphi_{12} - 2\varphi_\pi)]\}. \qquad (12)$$

The coefficients f_i and g_i are given in the Appendix. Equation (11), unlike the differential cross section for inelastic processes in which two particles are ejected, contains asymmetries with respect to the relative azimuth $(\varphi_\pi - \varphi_{12})$ in the forms of $\cos(\varphi_\pi - \varphi_{12})$ and $\cos 2(\varphi_\pi - \varphi_{12})$. Both asymmetries must, of course, be measured to determine the two independent coefficients f_4 and f_5. The kinematic scheme of this experiment in the c.m.s. of the colliding nucleons is represented in Fig. 2. The relation between the radius R and the π-meson momentum in this diagram corresponds to a π-meson energy which is close to the maximum. For fixed angles θ_π and θ_{12}, as well as for $\varphi_\pi = 0$, the cross section (11) becomes

$$\langle T_{00}\rangle_{\text{unpol}} = h_0 + h_1\cos\varphi_{12} + h_2\cos 2\varphi_{12}.$$

To obtain the coefficients h_1 and h_2 (and thus f_4 and f_5), measurements must be made at the angles $\varphi_{12} = 0$, 90, and 180°. In measuring the asymmetry associated with $\cos 2(\varphi_\pi - \varphi_{12})$ it is expedient to have the angles θ_π and θ_{12} equal 90°, in which case the term containing $\cos(\varphi_\pi - \varphi_{12})$ disappears.

Measurement of f_5 is very important because this coefficient depends only on the transitions which are associated with the initial states $^3P_0$, $^3P_2$, and $^3F_2$ of two protons and involves no other transition. The following general conclusion is evidently justified: the term containing f_5 receives contributions from those triplet states of the nucleon, in which the orbital and spin angular momenta are either parallel or antiparallel.

As an example we can consider the reaction $p + p \rightarrow \pi^0 + p + p$, to which only σ_{11} transitions can contribute. In our particular selection of transitions the measurement of f_5' directly gives

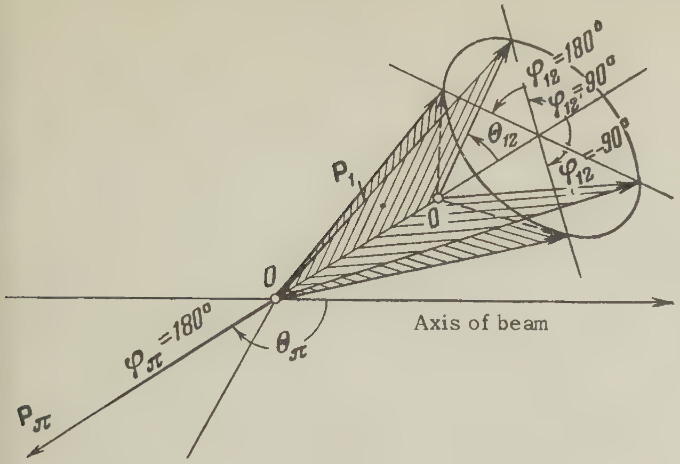

FIG. 2

the amplitude of the transition $^3P_0 \rightarrow (^3P_1p)_0$. In this case we have

$$f'_5 = -{}^3/_8\,|d_1|^2. \qquad (13)$$

The minus sign in (13) indicates that emission is more probable when $(\varphi_\pi - \varphi_{12})$ equals 90° than when it equals 0°.

Experiments in which only the π meson is recorded and the nucleons can be ejected in any directions, give less information than when one of the nucleons is also recorded. Indeed, when only the π meson is observed we have

$$\langle T_{00}\rangle_{\text{unpol}}(\theta_\pi,\ \varphi_\pi;\ p_\pi) = (f_0 + {}^1/_3 f_2) + (f_1 + {}^1/_3 f_3)\cos^2\theta_\pi, \qquad (14)$$

where the coefficients f_4 and f_5 have disappeared completely and the remaining coefficients occur in pairs.

From consideration of the coefficients which determine $<T_{00}>_{\text{pol}}$ it is seen that g_1, g_7, g_8, and g_9 contain terms that result from the interference of σ_{10}- and σ_{11}-transitions, respectively: namely between the transitions

$$^1S_0 \rightarrow (^3S_1p)_0 \text{ and } ^1D_2 \rightarrow (^3S_1p)_2$$

and the transitions

$$^3P_1 \rightarrow (^3P_0p)_1, \quad ^3P_1 \rightarrow (^3P_1p)_1, \quad ^3P_1 \rightarrow (^3P_2p)_1.$$

Experiments with polarized beams when only π mesons are recorded enable us to determine

$$\langle T_{00}\rangle_{\text{pol}}\ (\theta_\pi,\ \varphi_\pi;\ p_\pi) = \sin\theta_\pi\cos\theta_\pi\left(g_0 + {}^1/_3 g_2\right)\cos\varphi_\pi, \qquad (15)$$

where seven of the nine coefficients have disappeared and the remaining two appear in linear combination. This is a convincing illustration of how much more information is obtained from experiments in which one of the nucleons is observed in addition to the π meson.

In conclusion the author wishes to thank S. M. Bilen'kii, L. I. Lapidus, and R. M. Ryndin for valuable discussions and suggestions.

APPENDIX*

The Coefficients f_i and g_i for the Reaction $p + p \rightarrow \pi^+ + n + p$

$$f_0 = \frac{1}{2}c_-^2 + \frac{9}{4}\Big\{\frac{1}{6}d_1^2 + d_2^2 + \frac{3}{4}d_3^2 + \frac{5}{4}d_4^2 + \frac{4}{\sqrt{5}}d_2d_4\cos(d_2,\ d_4) - \frac{3}{2}\sqrt{\frac{3}{5}}d_3d_4\cos(d_3,\ d_4) + \frac{1}{3}(b_1^2 + b_3^2) + \frac{1}{2}b_4^2$$
$$+ \sqrt{\frac{2}{3}}b_3b_4\cos(b_3,\ b_4) - \sqrt{\frac{2}{3}}b_1b_4\cos(b_1,\ b_4) - \frac{2}{3}b_1b_3\cos(b_1,\ b_3)\Big\};$$

$$f_1 = \frac{c_+^2 - c_-^2}{2} + \frac{9}{4}\Big\{\frac{1}{6}d_1^2 - d_2^2 - \frac{3}{4}d_4^2 - \frac{4}{\sqrt{5}}d_2d_4\cos(d_2,\ d_4) - \frac{1}{3}b_1^2 + \frac{7}{6}b_3^2 + \frac{3}{2}b_2^2 + \frac{1}{2}b_4^2 - 4\sqrt{\frac{3}{2}}b_3b_4\cos(b_3,\ b_4)$$
$$+ \sqrt{\frac{2}{3}}b_1b_4\cos(b_1,\ b_4) + \frac{2}{3}b_1b_3\cos(b_1,\ b_3) - 2\sqrt{\frac{3}{2}}b_2b_4\cos(b_2,\ b_4) + 3b_2b_3\cos(b_2,\ b_3)\Big\};$$

$$f_2 = \frac{9}{4}\Big[\frac{1}{6}d_1^2 - \frac{3}{4}d_3^2 - \frac{21}{20}d_4^2 - \frac{6}{\sqrt{5}}d_2d_4\cos(d_2,\ d_4) + \frac{3}{2}\sqrt{\frac{3}{5}}d_3d_4\cos(d_3,\ d_4) - \frac{1}{3}b_1^2 + \frac{3}{2}b_2^2 + \frac{7}{6}b_3^2 + \frac{1}{2}b_4^2$$
$$+ \sqrt{\frac{2}{3}}b_1b_4\cos(b_1,\ b_4) + \frac{2}{3}b_1b_3\cos(b_1,\ b_3) - 4\sqrt{\frac{2}{3}}b_3b_4\cos(b_3,\ b_4) + \sqrt{6}\,b_2b_4\cos(b_2,\ b_4) - 3b_2b_3\cos(b_2,b_3)\Big];$$

$$f_3 = \frac{9}{4}\Big[-\frac{1}{2}d_1^2 + \frac{3}{2}d_3^2 + \frac{3}{2}d_4^2 + \frac{6}{\sqrt{5}}d_2d_4\cos(d_2,\ d_4) - 6\sqrt{\frac{3}{5}}d_3d_4\cos(d_3,\ d_4) + b_1^2 - 3b_2^2 - \frac{1}{2}b_4^2$$
$$+ 5\sqrt{6}\,b_3b_4\cos(b_3,\ b_4) + \sqrt{6}\,b_1b_4\cos(b_1,\ b_4) + 2b_1b_3\cos(b_1,\ b_3)\Big];$$

$$f_4 = 3\Big[-\frac{1}{2}d_1^2 - \frac{63}{80}d_4^2 - \frac{9}{8}d_3^2 + \frac{3}{4}\sqrt{3}\,d_2d_3\cos(d_2,\ d_3) + \frac{3}{4}\sqrt{15}\,d_3d_4\cos(d_3,\ d_4) - \frac{3}{4}\sqrt{\frac{1}{5}}d_2d_4\cos(d_2,\ d_4)$$
$$+ b_1^2 - \frac{9}{4}b_2^2 + \frac{1}{3}b_3^2 - \frac{3}{2}b_4^2 + \sqrt{\frac{3}{2}}b_1b_4\cos(b_1,\ b_4) + b_1b_3\cos(b_1,\ b_3) - \frac{21}{\sqrt{6}}b_3b_4\cos(b_3,b_4)\Big];$$

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*In all formulas of the Appendix the letters b, c and d denote absolute values.

$$f_5 = \frac{3}{2}\left[-\frac{1}{4}d_1^2 + \frac{1}{2}(b_1^2 + b_3^2) + \frac{3}{4}b_4^2 + \sqrt{\frac{3}{2}}\,b_3b_4\cos(b_3, b_4) - \sqrt{\frac{3}{2}}\,b_1b_4\cos(b_1, b_4) - b_1b_3\cos(b_1, b_3)\right];$$

and also

$$g_0 = \frac{9}{4}\left[\frac{1}{2}d_1d_3\sin(d_1, d_3) - \frac{1}{\sqrt{20}}d_1d_4\sin(d_1, d_4) - b_1b_2\sin(b_1, b_2) - b_1b_3\sin(b_1, b_3) + \frac{1}{\sqrt{3}}b_1b_4\sin(b_1, b_4)\right.$$
$$\left. - b_2b_3\sin(b_2, b_3) - \sqrt{\frac{3}{2}}\,b_2b_4\sin(b_2, b_4) - \frac{5}{\sqrt{6}}b_3b_4\sin(b_3, b_4)\right];$$

$$g_1 = \frac{3}{2}\left[\sqrt{\frac{3}{2}}\,c_-d_3\cos(c_-, d_3) - \frac{3}{\sqrt{10}}c_-d_4\cos(c_-, d_4) - c_+d_2\cos(c_+, d_2) + \frac{1}{\sqrt{5}}c_+d_4\cos(c_+, d_4)\right];$$

$$g_2 = \frac{9}{4}\left[\frac{1}{2}d_1d_3\sin(d_1, d_3) - \frac{1}{2}\sqrt{\frac{3}{5}}\,d_1d_4\sin(d_1, d_4) + 3b_1b_2\sin(b_1, b_2) - b_1b_3\sin(b_1, b_3) + b_1b_4\sin(b_1, b_4)\right.$$
$$\left. + 5b_2b_3\sin(b_2, b_3) - 3\sqrt{\frac{3}{2}}\,b_2b_4\sin(b_2, b_4) + \frac{25}{\sqrt{6}}b_3b_4\sin(b_3, b_4)\right];$$

$$g_3 = \frac{9}{4}\left[\frac{1}{2}d_1d_3\sin(d_1, d_3) - \frac{1}{2}\sqrt{\frac{3}{5}}\,d_1d_4\sin(d_1, d_4) - b_1b_2\sin(b_1, b_2) - b_1b_3\sin(b_1, b_3) + \sqrt{\frac{2}{3}}\,b_1b_4\sin(b_1, b_4)\right.$$
$$\left. - b_2b_3\sin(b_2, b_3) - \sqrt{\frac{3}{2}}\,b_2b_4\sin(b_2, b_4) - \frac{5}{\sqrt{6}}b_3b_4\sin(b_3, b_4)\right];$$

$$g_4 = \frac{9}{4}\left[-\frac{1}{2}d_1d_3\sin(d_1, d_3) + \frac{1}{2}\sqrt{\frac{3}{5}}\,d_1d_4\sin(d_1, d_4) + b_1b_2\sin(b_1, b_2) - b_1b_3\sin(b_1, b_3) + \sqrt{\frac{2}{3}}\,b_1b_4\sin(b_1, b_4)\right.$$
$$\left. + b_2b_3\sin(b_2, b_3) + \sqrt{\frac{3}{2}}\,b_2b_4\sin(b_2, b_4) - \frac{5}{\sqrt{6}}b_3b_4\sin(b_3, b_4)\right];$$

$$g_5 = \frac{9}{4}\left[-\frac{1}{2}d_1d_3\sin(d_1, d_3) + \frac{1}{2}\sqrt{\frac{3}{5}}\,d_1d_4\sin(d_1, d_1) - 3b_1b_2\sin(b_1, b_2) - b_1b_3\sin(b_1, b_3) + \sqrt{\frac{2}{3}}\,b_1b_4\sin(b_1, b_4)\right.$$
$$\left. + 3b_2b_3\sin(b_2, b_3) + 3\sqrt{\frac{3}{2}}\,b_2b_4\sin(b_2, b_4) + \frac{25}{\sqrt{6}}b_3b_4\sin(b_3, b_4)\right];$$

$$g_6 = \frac{9}{4}\left[-\frac{1}{\sqrt{3}}d_1d_3\sin(d_1, d_3) + \frac{1}{2}\sqrt{\frac{3}{5}}\,d_1d_4\sin(d_1, d_4) + b_1b_2\sin(b_1, b_2) - b_1b_3\sin(b_1, b_3) + \sqrt{\frac{2}{3}}\,b_1b_4\sin(b_1, b_4)\right.$$
$$\left. + b_2b_3\sin(b_2, b_3) - \frac{5}{\sqrt{6}}b_3b_4\sin(b_3, b_4) + \sqrt{\frac{3}{2}}\,b_2b_4\sin(b_2, b_4)\right];$$

$$g_7 = \frac{9}{4}\left[-\frac{1}{3\sqrt{2}}c_-d_2\cos(c_-, d_2) - \frac{1}{\sqrt{6}}c_-d_3\cos(c_-, d_3) - \frac{7}{3}\sqrt{\frac{1}{10}}\,c_-d_4\cos(c_-, d_4)\right];$$

$$g_8 = \frac{9}{4}\left[\frac{7}{3}\sqrt{\frac{1}{10}}\,c_-d_4\cos(c_-, d_4) + \frac{\sqrt{2}}{3}c_-d_2\cos(c_-, d_2) + \frac{1}{\sqrt{6}}c_-d_3\cos(c_-, d_3)\right.$$
$$\left. - \frac{1}{\sqrt{3}}c_+d_3\cos(c_+, d_3) - \frac{1}{\sqrt{5}}c_+d_4\cos(c_+, d_4)\right];$$

$$g_9 = \frac{3}{2}\left[\frac{1}{2}\sqrt{\frac{3}{2}}\,c_-d_3\cos(c_-, d_3) - \frac{1}{\sqrt{2}}c_-d_2\cos(c_-, d_2) - \frac{1}{2\sqrt{10}}c_-d_4\cos(c_-, d_4)\right].$$

The Coefficients f_i' and g_i' for the Process $p + p \to \pi^0 + p + p$

$$f_0' = \frac{9}{4}\left[\frac{1}{6}d_1^2 + d_2^2 + \frac{3}{4}d_4^2 + \frac{5}{4}d_4^2 + \frac{4}{\sqrt{5}}d_2d_4\cos(d_2, d_4) - \frac{3}{5}\sqrt{\frac{3}{5}}\,d_3d_4\cos(d_3, d_4)\right];$$

$$f_1' = \frac{9}{4}\left[\frac{1}{6}d_1^2 - d_2^2 - \frac{3}{4}d_3^2 - \frac{4}{\sqrt{5}}d_2d_4\cos(d_2, d_4)\right];$$

$$f_2' = \frac{9}{4}\left[\frac{1}{6}d_1^2 - \frac{3}{4}d_3^2 - \frac{21}{20}d_4^2 - \frac{6}{\sqrt{5}}d_2d_4\cos(d_2, d_4) + \frac{3}{2}\sqrt{\frac{3}{5}}\,d_3d_4\cos(d_3, d_4)\right];$$

$$f_3' = \frac{9}{4}\left[-\frac{1}{2}d_1^2 + \frac{3}{2}d_3^2 + \frac{3}{2}d_4^2 + \frac{6}{\sqrt{5}}d_2d_4\cos(d_2, d_4) - 6\sqrt{\frac{3}{5}}\,d_3d_4\cos(d_3, d_4)\right];$$

$$f_4' = 3\left[-\frac{1}{2}d_1^2 - \frac{63}{80}d_4^2 - \frac{9}{8}d_3^2 + \frac{3}{4}\sqrt{3}\,d_2d_3\cos(d_2, d_3) + \frac{3}{4}\sqrt{15}\,d_3d_4\cos(d_3, d_4)\right];$$

$$f_5' = -\frac{3}{8}d_1^2;$$

and also

$$g'_0 = \frac{9}{4}\left[\frac{1}{2} d_1 d_3 \sin(d_1, d_3) - \frac{1}{\sqrt{20}} d_1 d_4 \sin(d_1, d_4)\right];$$

$$g'_2 = \frac{9}{4}\left[\frac{1}{2} d_1 d_3 \sin(d_1, d_3) - \frac{1}{2}\sqrt{\frac{3}{5}} d_1 d_4 \sin(d_1, d_4)\right];$$

$$g'_6 = \frac{9}{4}\left[-\frac{1}{\sqrt{3}} d_1 d_3 \sin(d_1, d_3) + \frac{1}{2}\sqrt{\frac{3}{5}} d_1 d_4 \sin(d_1, d_4)\right];$$

$$g'_3 = g'_2 = -g'_4 = -g'_5;$$

$$g'_1 = g'_7 = g'_8 = g'_9 = 0.$$

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EXTENSION OF THE BOGOLIUBOV-TIABLIKOV PERTURBATION METHOD TO A NON-STATIONARY CASE

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The perturbation theory in the second-quantization representation, developed by Bogoliubov and Tiablikov in the stationary case for use with the polar model of crystals, is generalized to include the case of perturbation by the electromagnetic field of a light wave. A general expression is derived for the deformed operator of the electric current density excited by the radiation perturbation. A possible application of the method in the theory of optical properties of cyrstals is indicated.

1. INTRODUCTION

THE perturbation-theory method of Bogoliubov and Tiablikov[1-3] is based on the introduction of an operator that projects an arbitrary wave function of the system of valence electrons of the crystal in the homopolar* functions of the problem. In this variant of the theory the mean value of the electric current produced in the crystal by a weak constant external field is non-vanishing only in third approximation. This makes the method unsuitable for the consideration of the electrical and optical properties of metals, and also for the consideration of the strongly excited "current" states of semiconductors within the framework of the polar model, although it does not diminish its usefulness for the description of the properties of crystals that are determined by exchange interactions, for example.

Nevertheless it seems to us that the Bogoliubov-Tiablikov method can be used for the treatment of the electrical and optical properties of electronic semiconductors at low temperatures. Here we have to do with weak electric fields and low frequencies of the optical spectrum. The point is that at low temperatures electronic semiconductors have practically the properties of insulators, which offers the possibility of confining ourselves in a certain approximation to the consideration of just the weakly polarized (quasi-homopolar) states of the crystal. Moreover the limitation of low frequencies ($\omega < \omega_0$, where ω_0 is the minimum angular frequency necessary for the appearance of "current" states) in the consideration of the optical phenomena assures the correctness of neglecting the effects of quantum transitions of the electronic system into the polar excited states of the semiconductor. Although the region of applicability of a method in which one considers only quasi-homopolar states of a crystal to the treatment of, for example, optical properties of crystals is very limited, it is well to begin the discussion of such properties in the framework of a many-electron theory with this simplest case. In the present paper we carry out the extension of the Bogoliubov-Tiablikov method to the non-stationary case of the alternating electromagnetic field of a light wave propagated in the crystal.

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*By the homopolar states in the polar model of a crystal we mean states in which there is always one valence electron close to each lattice point of the crystal.

2. THE HAMILTONIAN OF THE SYSTEM OF ELECTRONS IN A CRYSTAL IN AN ELECTROMAGNETIC FIELD, IN THE SECOND-QUANTIZATION REPRESENTATION

Let us consider a simple atomic cubic crystal, in which there is at each lattice point a single electron, which is an s state. In what follows we shall neglect the thermal vibrations of the lattice ions.

As in Refs. 1—3, let us denote by the symbol $f = (f^1f^2f^3)$ the coordinates of any lattice point, and use the lattice constant a as unit of length. We denote the position coordinates of the j-th electron by $\mathbf{r}_j = (x_jy_jz_j)$ and its spin coordinate by s_j.

The atomic wave functions $\varphi_{f\sigma}(\mathbf{r}; s) = \varphi_f(\mathbf{r}) \times \delta(s - \sigma)$ (where $\sigma = \pm\frac{1}{2}$ is the spin quantum number) at two different lattice points are not strictly orthogonal, so that in the general case the system of functions $\varphi_{f\sigma}$ is not an orthogonal system and a rigorous application of the method of second quantization is not possible. It is always possible, however, as shown in Ref. 3, to introduce an equivalent system of new "quasi-atomic" functions $\Theta_{f\sigma}$ which are exactly orthogonal to each other

$$\Theta_{f\sigma}(\mathbf{r}; s) = \Theta_f(\mathbf{r})\,\delta(s-\sigma). \tag{2.1}$$

This makes it possible to apply the method of second quantization.

Suppose that an electromagnetic wave of frequency ω is propagated through the crystal. The vector potential $\mathbf{A}$ of this wave can always be chosen so that

$$\operatorname{div}\mathbf{A} = 0, \tag{2.2}$$

and the scalar potential can be takn to be zero.[4]

Neglecting quantities of order A^2 and using Eq. (2.2), we can write the Hamiltonian in the second-quantization representation for a system of electrons in a crystal in an external electromagnetic field in the form (cf. Refs. 3 and 5)

$$\hat{H}(t) = \hat{H} + \hat{W}(t), \tag{2.3}$$

where

$$\hat{H} = \hat{H}_0 + \varepsilon\hat{H}_1 + \varepsilon^2\hat{H}_2, \tag{2.4}$$

$$\hat{W}(t) = \sum_{ff'\sigma\sigma'} W(ff';\, t)\,\hat{a}^+_{f\sigma}\hat{a}_{f'\sigma'}, \tag{2.5}$$

$$W(ff',\, t) = \int \Theta_f(\mathbf{r})\left[-\frac{ie\hbar}{mc}\mathbf{A}(\mathbf{r};\, t)\,\nabla\right]\Theta_{f'}(\mathbf{r})\,d\mathbf{r}, \tag{2.6}$$

$\hat{a}$ and $\hat{a}^+$ are the Fermi operators of second quantization, e is the absolute value of the electronic charge, and m is the mass of the electron.

The quantity ϵ in Eq. (2.4) is a small parameter (the non-orthogonality integral) with respect to which the Hamiltonian is expanded [see Ref. 3, Eq. (4.63)]. It must be kept in mind that each of the quantities $\hat{H}_0$, $\epsilon\hat{H}_1$, and $\epsilon^2\hat{H}_2$ can also be expanded as a power series in ϵ. The way Eq. (2.4) is written means that such an expansion of $\hat{H}_0$ begins with a term of zeroth order of smallness, that of $\epsilon\hat{H}_1$ with a first order term, and so on, although each of these expansions contains also terms of higher orders.

In the second-quantization representation the Schrödinger equation for the system of electrons takes the form

$$i\hbar\,\partial C(\ldots n_{f\sigma}\ldots;\, t)/\partial t = \hat{H}(t)\,C(\ldots n_{f\sigma}\ldots;\, t), \tag{2.7}$$

where $C(\ldots n_{f\sigma}\ldots;\, t) = C(n;\, t)$ is the wave function of the system and depends on the numbers of filled "quasi-atomic" states $\Theta_{f\sigma}$.

We assume as usual that for the visible and infrared regions the wavelength of the electromagnetic radiation is much larger than the dimensions of the "basic" domain in the crystal, so that the dependence of the vector potential on the coordinates can be neglected. Then Eq. (2.6) can be put in the form

$$W(ff';\, t) = \frac{e}{mc}\mathbf{A}(t)\,\mathbf{I}(ff'), \tag{2.8}$$

where $\mathbf{I}(ff')$ is as defined in Ref. 3 and is of first order in ϵ [cf. Eq. (4.41) and page 134 in Ref. 3], and

$$\mathbf{A}(t) = \mathbf{B}e^{i\omega t} + \mathbf{B}^* e^{-i\omega t}. \qquad (2.9)$$

Therefore Eq. (2.5) can be written in the following form:

$$\hat{W}(t) = \varepsilon \hat{H}_1'(t) = \frac{e}{mc}(\mathbf{A}(t)\,\mathbf{I}), \qquad (2.10)$$

where **I** is the operator of the total momentum of the system in the second-quantization representation [cf. Eq. (4.40) in Ref. 3].

Thus for the case considered Eq. (2.7) is written as follows:

$$(\hat{H}_0 + \varepsilon\hat{\mathscr{H}}_1(t) + \varepsilon^2\hat{H}_2 - i\hbar\partial/\partial t)\,C = 0, \qquad (2.11)$$

where we have introduced the notation

$$\hat{\mathscr{H}}_1(t) = \hat{H}_1 + \hat{H}_1'(t). \qquad (2.12)$$

3. DEFINITION AND PROPERTIES OF THE OPERATOR OF PROJECTION ONTO THE L-SPACE

In zeroth approximation Eq. (2.11) takes the form

$$(\hat{H}_0 - i\hbar\partial/\partial t)\,C = 0 \qquad (3.1)$$

with the stationary solution

$$C(n;\ t) = C(n)\exp\left[-\frac{i}{\hbar}Et\right], \qquad (3.2)$$

given that

$$\hat{H}_0 C(n) = EC(n). \qquad (3.3)$$

The eigenvalues E in Eq. (3.3) depend on the total occupation numbers N_f^0:

$$E = E(\cdots N_f^0 \cdots),\ N_f = \sum_\sigma n_{f\sigma} \qquad (3.4)$$

[cf. Eq. (4.69) in Ref. 3]. The eigenfunctions corresponding to Eq. (3.4) are written in the form

$$C_\psi(n) = \Psi(n)\prod_f \delta(N_f - N_f^0), \qquad (3.5)$$

where Ψ is an arbitrary function of the occupation numbers. Thus in zeroth order the energy levels of the system are degenerate, and in the case in question the part of the "index" of the degenerate state is played by the function Ψ itself, since the form of the eigenfunction C_ψ depends on the choice of this function. We shall assume that the system of the C_ψ is already orthonormal.

Bogoliubov[3] has shown that in the approximation of quasi-homopolarity the lowest eigenvalue of the operator $\hat{H}_0$ is given by

$$E_0 = E(\cdots 1 \cdots). \qquad (3.6)$$

Following Refs. 1–3, we shall call the subspace (manifold) of eigenfunctions

$$C_{0\psi} = \Psi(n)\prod_f \delta(N_f - 1), \qquad (3.7)$$

corresponding to the lowest eigenvalue E_0 (the exactly homopolar states) the L-space.

We introduce the operator $\hat{P}_\psi$ which projects an arbitrary wave function C (n; t) onto the axis $C_{0\psi}$ of the L-space:

$$\hat{P}_\psi C(n;\ t) = a_\psi(t)\, C_{0\psi}(n), \qquad (3.8)$$

where $a_\psi(t)$ is a coefficient independent of the variables n but in general dependent on the time t. We determine this coefficient from the condition that the difference between the function C (n; t) and its projection onto the axis $C_{0\psi}$ is orthogonal to this axis:

$$(C_{0\psi}^*[C - \hat{P}_\psi C]) = 0. \qquad (3.9)$$

From this, using Eq. (3.8) and the orthonormality of the $C_{0\psi}$, we get

$$a_\psi(t) = (C_{0\psi}^* C). \qquad (3.10)$$

We call the operator

$$\hat{P} = {\sum_\psi}' \hat{P}_\psi \qquad (3.11)$$

the projection operator onto the entire L-space. Clearly we have the equation

$$\hat{P}C(n;\ t) = \sum_\psi a_\psi(t)\, C_{0\psi}(n), \qquad (3.12)$$

where $a_\psi(t)$ is defined by Eq. (3.10)

It is also not difficult to show that the projection operator has the following properties:

$$\hat{P}^2 = \hat{P},\ \hat{P}^+ = \hat{P},\ \hat{P}\frac{\partial}{\partial t} = \frac{\partial}{\partial t}\hat{P}; \qquad (3.13)$$

$$\hat{P}\hat{H}_0 = \hat{H}_0\hat{P} = E_0\hat{P}; \qquad (3.14)$$

$$\hat{P}(\hat{H}_0 - i\hbar\partial/\partial t) = (\hat{H}_0 - i\hbar\partial/\partial t)\,\hat{P},\ \hat{P}(\hat{H}_0 - i\hbar\partial/\partial t)^{-1} = (\hat{H}_0 - i\hbar\partial/\partial t)^{-1}\hat{P}. \qquad (3.15)$$

The last equation is valid only in cases in which the inverse operator contained in it actually exists (which is true in our further considerations).

4. EXTENSION OF THE BOGOLIUBOV-TIABLIKOV PERTURBATION METHOD TO THE NON-STATIONARY CASE

An arbitrary function C (n; t) can be represented in the form

$$C = C_0 + C_1, \qquad (4.1)$$

where $C_0 = \hat{P}C$ is its projection onto the L-space and $C_1 = (1 - \hat{P})C$ is a function orthogonal to the L-space, i.e.,

$$\hat{P}C_1 = 0. \tag{4.2}$$

Substituting Eq. (4.1) into Eq. (2.11), multiplying the resulting equation from the left by the operator $\hat{P}$, and using the properties of this operator, we get:

$$(\hat{H}_0 + \varepsilon\hat{P}\hat{\mathscr{H}}_1(t)\hat{P} + \varepsilon^2\hat{P}\hat{H}_2\hat{P} - i\hbar\partial/\partial t)\hat{P}C$$
$$+ (\varepsilon\hat{P}\hat{\mathscr{H}}_1(t) + \varepsilon^2\hat{P}\hat{H}_2)C_1 = 0. \tag{4.3}$$

Subtracting the two members of this equation from the equation which we had before multiplying by the operator $\hat{P}$, we find:

$$(\varepsilon\hat{\mathscr{H}}_1(t) + \varepsilon^2\hat{H}_2 - \varepsilon^2\hat{P}\hat{\mathscr{H}}_1(t)\hat{P} - \varepsilon^2\hat{P}\hat{H}_2\hat{P})\hat{P}C$$
$$+ (\hat{H}_0 + \varepsilon\hat{\mathscr{H}}_1(t) + \varepsilon^2\hat{H}_2 - \varepsilon\hat{P}\hat{\mathscr{H}}_1(t) \tag{4.4}$$
$$- \varepsilon\hat{P}\hat{H}_2 - i\hbar\partial/\partial t)C_1 = 0.$$

Since by Eq. (2.10) the perturbation by the light wave is of first order in ϵ, we can try to evaluate the function C_1 of Eq. (4.1) in the form of the series

$$C_1(n;\ t) = \varepsilon K(n;\ t) + \varepsilon^2 L(n;\ t) + \ldots \tag{4.5}$$

Substituting Eq. (4.5) into Eq. (4.4), we get equations for the functions K and L; the solutions can be written in the form

$$K(n;\ t) = (\hat{H}_0 - i\hbar\partial/\partial t)^{-1}(\hat{P}\hat{\mathscr{H}}_1(t)\hat{P} - \hat{\mathscr{H}}_1(t))\hat{P}C(n;\ t),$$

$$L(n;\ t) = (\hat{H}_0 - i\hbar\partial/\partial t)^{-1}(\hat{P}\hat{H}_2\hat{P} - \hat{H}_2)\hat{P}C(n;\ t) + (H_0 - i\hbar\partial/\partial t)^{-1} \tag{4.6}$$
$$\times (\hat{P}\hat{\mathscr{H}}_1(t) - \hat{\mathscr{H}}_1(t))(\hat{H}_0 - i\hbar\partial/\partial t)^{-1}(\hat{P}\hat{\mathscr{H}}_1(t)\hat{P} - \hat{\mathscr{H}}_1(t))\hat{P}C(n;\ t).$$

Generally speaking, the operator $(\hat{H}_0 - i\hbar\partial/\partial t)^{-1}$ does not always have a meaning. But in what follows we need only to know its effect on functions that depend exponentially on the time. In this case

$$(\hat{H}_0 - i\hbar\partial/\partial t)^{-1}e^{-i\Omega t} = e^{-i\Omega t}(H_0 - \hbar\Omega)^{-1}. \tag{4.7}$$

Since in fact $\hat{H}_0$ is not an operator, but a c-number, this expression always has a meaning except in the case $H_0 = \hbar\Omega$, which, as we shall see below, corresponds to a resonance. Consequently the calculations presented here hold for frequencies of the light that do not coincide with the proper frequencies of the system. Inclusion of damping makes it possible to consider the general case of arbitrary frequency of the incident light, as will be shown in a subsequent paper.

Using the properties of the projection operator (cf. Sec. 3), we can show without difficulty that

$$\hat{P}K = 0,\quad \hat{P}L = 0. \tag{4.8}$$

Substituting (4.6) into (4.5), and then (4.5) into (4.3), and confining ourselves to the third approximation, we get

$$i\hbar\partial C_0(n;\ t)/\partial t = \tilde{H}(t)C_0(n;\ t), \tag{4.9}$$

where $\tilde{H}(t)$ differs from the corresponding expression for $\tilde{H}$ in Ref. 3 (cf. pp. 179 and 147) by the replacement of $\hat{H}_1$ by $\mathscr{H}_1(t)$ and E by $i\hbar \times \partial/\partial t$.

Using the properties of the operator $\hat{P}$ and of the Fermi operators, one can prove the following relations:

$$\hat{P}\varepsilon\hat{H}_1\hat{P} = 0,\quad \hat{P}\varepsilon\hat{H}_1'(t)\hat{P} = 0. \tag{4.10}$$

Recalling Eq. (2.12), we can separate the operator $\tilde{H}(t)$ into a term depending on the radiation field and one independent of it. Keeping only terms linear in the radiation field amplitude and noting the relations (4.10), we represent $\tilde{H}(t)$ in the form:

$$\tilde{H}(t) = \tilde{H} + \Delta\tilde{H} + \tilde{W}(t), \tag{4.11}$$

where $\tilde{H}$ and $\Delta\tilde{H}$ agree with the corresponding notations in Ref. 3 (see page 179), differing from them only by the replacement of E_0 by $i\hbar\partial/\partial t$, and

$$\tilde{W}(t) = \varepsilon^3\hat{P}\{-\hat{H}_1'(t)(H_0 - i\hbar\partial/\partial t)^{-1}\hat{H}_2 - \hat{H}_2$$
$$\times(H_0' - i\hbar\partial/\partial t)^{-1}\hat{H}_1'(t)$$
$$+ \hat{H}_1'(t)(H_0 - i\hbar\partial/\partial t)^{-1}\hat{H}_1(H_0 - i\hbar\partial/\partial t)^{-1}\hat{H}_1$$
$$+ \hat{H}_1(H_0 - i\hbar\partial/\partial t)^{-1}\hat{H}_1'(t)(H_0 - i\hbar\partial/\partial t)^{-1}\hat{H}_1 \tag{4.12}$$
$$+ \hat{H}_1(H_0 - i\hbar\partial/\partial t)^{-1}\hat{H}_1(H_0 - i\hbar\partial/\partial t)^{-1}\hat{H}_1'(t)\}\hat{P}.$$

In obtaining Eq. (4.12) we have used the relations

$$\varepsilon^2\hat{P}\hat{H}_1'(t)(H_0 - i\hbar\partial/\partial t)^{-1}\hat{H}_1\hat{P} = 0;$$
$$\varepsilon^2\hat{P}\hat{H}_1(H_0 - i\hbar\partial/\partial t)^{-1}\hat{H}_1'(t)\hat{P} = 0, \tag{4.13}$$

which can be proved without difficulty by using the symmetry properties of the matrix elements appearing in the expressions for $\hat{H}_1$ and $\hat{H}_1'(t)$.

Equation (4.9) is now written in the form

$$i\hbar \partial C_0 / \partial t = [\tilde{H} + \Delta\tilde{H} + \tilde{W}(t)]\, C_0. \tag{4.14}$$

The solution of this equation has the form

$$C_0(n;\ t) = C_0^0(n) \exp\left(-\frac{i}{\hbar} Et\right) + C_0'(n;\ t), \tag{4.15}$$

where the first term corresponds to the solution of the stationary equation [with $\tilde{W}(t) = 0$] and C_0' is the part determined by the radiative perturbation. Since the operator $\tilde{W}(t)$ is of third order in ϵ, C_0' will also be of third order. According to Ref. 3 the energy E in Eq. (4.15) is given by

$$E = E_0 + \varepsilon^2 E_1 + \varepsilon^3 E_2 + \cdots, \tag{4.16}$$

The first order term is absent from this expression, since in first approximation the Hamiltonian $\tilde{H} + \Delta\tilde{H}$ does not differ from the same operator in zeroth approximation.[3]

In the expression for the average value of the electric current of the system in the electromagnetic field

$$\mathbf{J} = \left(C^{\bullet} \left| -\frac{e}{m}\, \hat{\mathbf{I}} - \frac{e^2 N}{mc} \mathbf{A}(t) \right| C\right) =$$

$$= \left(C_0^{\bullet}(n) \exp\left[\frac{i}{\hbar} Et\right] + C_0^{\bullet\prime}(n;\ t) + C_1^{\bullet}(n;\ t) \right| -\frac{e}{m}\, \hat{\mathbf{I}}$$

$$-\frac{e^2 N}{mc} \mathbf{A}(t) \left| C_0(n) \exp\left[-\frac{i}{\hbar} Et\right] + C_0'(n;\ t) + C_1(n;\ t)\right) \tag{4.17}$$

the terms containing C_0' and $\hat{\mathbf{I}}$ are of order ϵ^4 or higher (since $C_0' \sim \epsilon^3$ and $\mathbf{I} \sim \epsilon$), and the terms containing C_0' and $\mathbf{A}(t)$ are quadratic in the radiation field amplitude; therefore in calculating the current to third order we can neglect the terms in question. Then, recalling Eq. (4.5), we have

$$\mathbf{J} = \left(C_0^{\bullet}(n) \exp\left[\frac{i}{\hbar} Et\right] + \varepsilon K^{\bullet} + \varepsilon^2 L^{\bullet} + \cdots\right.$$

$$\left.\cdots \left| -\frac{e}{m}\, \hat{\mathbf{I}} \right| C_0(n) \exp\left[-\frac{i}{\hbar} Et\right] + \varepsilon K + \varepsilon^2 L + \cdots\right). \tag{4.18}$$

In the expression for the current operator in Eq. (4.17) we consider only the first term $-(e/m)\hat{\mathbf{I}}$, since the second term is not actually an operator and gives a trivial correction to the dielectric permeability of the crystal.

Using Eq. (4.7) and the identity

$$i\hbar \frac{\partial}{\partial t} C_0(n) \exp\left[-\frac{i}{\hbar} Et\right] = \exp\left[+\frac{i}{\hbar} Et\right]\left(E + i\hbar \frac{\partial}{\partial t}\right) C_0(n),$$

we can put Eq. (4.18) in the following form:

$$\mathbf{J} = -\frac{e}{m} (C_0^{\bullet}(n)\, M(\hat{\mathbf{I}})\, C_0(n)), \tag{4.19}$$

where

$$M(\hat{\mathbf{I}}) = \hat{P}\hat{\mathbf{I}}\hat{P} + D(\hat{\mathbf{I}}) + D^{+}(\hat{\mathbf{I}}) = D(\hat{\mathbf{I}}) + D^{+}(\hat{\mathbf{I}}), \tag{4.20}$$

since $\hat{\mathbf{P}}\hat{\mathbf{I}}\hat{\mathbf{P}} = 0$, and

$$D(\hat{\mathbf{I}}) = \hat{P}\Big\{-\varepsilon\hat{\mathbf{I}}\left(H_0 - E_0 - i\hbar\frac{\partial}{\partial t}\right)^{-1} \hat{\mathscr{H}}_1(t) - \varepsilon^2 \hat{\mathbf{I}}\left(H_0 - E_0 - i\hbar\frac{\partial}{\partial t}\right)^{-1} (\hat{H}_2 - \hat{P}\hat{H}_2\hat{P}) + \varepsilon^2 \hat{\mathbf{I}}\left(H_0 - E_0 - i\hbar\frac{\partial}{\partial t}\right)^{-1} (\hat{\mathscr{H}}_1(t)$$

$$- \hat{P}\hat{\mathscr{H}}_1(t))\left(H_0 - E_0 - i\hbar\frac{\partial}{\partial t}\right)^{-1} \hat{\mathscr{H}}_1(t) + \frac{\varepsilon^2}{2}\left[\left(H_0 - E_0 - i\hbar\frac{\partial}{\partial t}\right)^{-1} \mathscr{H}_1(t)\right]^{+} \hat{\mathbf{I}}\left[\left(H_0 - E_0 - i\hbar\frac{\partial}{\partial t}\right)^{-1} \hat{\mathscr{H}}_1(t)\right]\Big\}\hat{P}. \tag{4.21}$$

In deriving this expression we have everywhere replaced E by E_0, in view of Eq. (4.16); that is, we have set

$$\varepsilon(H_0 - E_0 - \varepsilon^2 E_1 - i\hbar\partial/\partial t)^{-1} \approx \varepsilon(H_0 - E_0 - i\hbar\partial/\partial t)^{-1},$$

which is legitimate if we neglect in this expression all terms beginning from the third order of small quantities (which corresponds to the fact that in Eq. (4.21) terms beginning from the third order are dropped, because $\hat{\mathbf{I}} \sim \epsilon$).

From this it follows that in the approximation considered the absorption spectrum of the system of electrons in the crystal is a discrete one, since in this approximation the splitting-up of the energy levels into bands has practically no effect. Physically this means that the "pairs" and "holes" that appear under the action of the radiation field remain in bound states, and the absorption of light by the crystal is not photoelectrically active absorption, i.e., does not lead to the appearance of current states. It is only this sort of absorption of light that can be treated by the projection method, with the definition of the projection operator that we have given above.

From Eq. (4.20) we note that the operator $M(\hat{\mathbf{I}})$, like $\hat{\mathbf{I}}$ itself, is Hermitian. We call it the "deformed" operator for the current (in analogy with the usage in Refs. 1–3).

Thus, to determine the average value of the current (4.1), with C_1 given by Eq. (4.5), one needs only to find the projection of the wave function onto the L-space and the "deformed" current operator.

5. DETERMINATION OF THE "DEFORMED" CURRENT OPERATOR

By the use of Eq. (2.12) we can make a separation of the expression (4.21) into two terms, as in Eq. (4.11); we thus obtain a current operator independent of the radiative perturbation (this part of the current is the same as given in Ref. 3 and is of no interest to us here), and a current operator dependent on the radiative perturbation. We give only the final result of the calculation of this second term, because of the cumbersome, though simple, manipulations required; for more detailed information the reader is referred to Ref. 5:

$$M(\hat{\mathbf{I}}) = \frac{e}{mc\hbar}\Big\{-\sum S_1(ff_1) + \sum S_5(ff_1f_2) + \sum S_1(ff_1)\,\hat{a}^+_{f\sigma}\hat{a}_{f\sigma_1}\hat{a}^+_{f_1\sigma_1}\hat{a}_{f_1\sigma} - \sum\left[S_5(ff_1f_2) + S_6(ff_2f_1) + S_5(f_2ff_1)\right]\hat{a}^+_{f\sigma}\hat{a}_{f\sigma_1}\hat{a}^+_{f_1\sigma_1}\hat{a}_{f_1\sigma}$$

$$+ \sum\left[S_7(ff_1f_2) + S_7(f_1ff_2)\right]\hat{a}^+_{f\sigma}\hat{a}_{f\sigma_2}\hat{a}^+_{f_1\sigma_1}\hat{a}_{f_1\sigma}\hat{a}^+_{f_2\sigma_2}\hat{a}^+_{f_2\sigma_1}\Big\}\mathbf{A}(t) + i\frac{e}{mc\hbar}\Big\{\sum\left[S'_{33}(ff_1f_2) - S'_{33}(f_1ff_2)\right]\hat{a}^+_{f\sigma}\hat{a}_{f\sigma_2}\hat{a}^+_{f_1\sigma_1}\hat{a}_{f_1\sigma}\hat{a}^+_{f_2\sigma_2}\hat{a}_{f_2\sigma_1}\Big\}\dot{\mathbf{A}}(t). \tag{5.1}$$

Here we have introduced the following notations*

$$\begin{aligned} S_5(ff_1f_2) &= S_2(ff_1f_2) + S_{31}(ff_1f_2) + S_4(ff_1f_2),\\ S_6(ff_1f_2) &= S_2(ff_1f_2) + S_{32}(ff_1f_2) + S_4(ff_1f_2),\\ S_7(ff_1f_2) &= S_2(ff_1f_2) + S_3(ff_1f_2) + S_4(ff_1f_2). \end{aligned} \tag{5.2}$$

$$\begin{aligned} S_1(ff_1) &= \frac{\omega(ff_1)}{\omega(ff_1)^2 - \omega^2}D(ff_1),\\ S_2(ff_1f_2) &= \frac{\omega(ff_1)\cdot\Lambda(ff_2)}{\hbar\omega(ff_2)\left[\omega(ff_1)^2 - \omega^2\right]}D(ff_1;\ f_1f_2),\\ S_3(ff_1f_2) &= \frac{\left[\omega(ff_1)\cdot\omega(f_1f_2) + \omega^2\right]\Lambda(ff_2)}{\hbar\left[\omega(ff_1)^2 - \omega^2\right]\left[\omega(f_1f_2)^2 - \omega^2\right]}D(ff_1;\ f_1f_2),\\ S_{31}(ff_1f_2) &= \frac{\left[\omega(ff_1)\cdot\omega(f_1f_2) + \omega^2\right]\Lambda_1(ff_2)}{\hbar\left[\omega(ff_1)^2 - \omega^2\right]\left[\omega(f_1f_2)^2 - \omega^2\right]}D(ff_1;\ f_1f_2),\\ S_{32}(ff_1f_2) &= \frac{V_{f_1f_2}(ff_2)\left[\omega(ff_1)\,\omega(f_1f_2) + \omega^2\right]}{\hbar\left[\omega(ff_1)^2 - \omega^2\right]\left[\omega(f_1f_2)^2 - \omega^2\right]}D(ff_1;\ f_1f_2),\\ S'_{33}(ff_1f_2) &= \frac{\left[\omega(ff_1) + \omega(f_1f_2)\right]\cdot\left[F(ff_1;\ f_2f_1) - F(ff_2;\ f_2f_2)\right]}{\hbar\left[\omega(ff_1)^2 - \omega^2\right]\left[\omega(f_1f_2)^2 - \omega^2\right]}D(ff_1;\ f_1f_2),\\ S_4(ff_1f_2) &= \frac{\omega(f_1f_2)\,\Lambda(ff_2)}{\hbar\omega(ff_2)\left[\omega(f_1f_2)^2 - \omega^2\right]}D(ff_1;\ f_1f_2). \end{aligned} \tag{5.3}$$

$$\omega(ff_1) = \Delta(ff_1)/\hbar, \tag{5.4}$$

$$\Lambda(ff_1) = L(ff_1) + \sum_{\substack{f''\\(\neq f)}} F(ff'';\ f_1f''),\quad V_{f_2f_1}(ff_2) = L(ff_2) + \sum_{f''} F(ff'';\ f_2f'')\left(1 - \delta_{f''f_2} + \delta_{f''f_1} - \delta_{f''f}\right). \tag{5.5}$$

In addition we have introduced the tensor notations (i, k = x, y, z):

$$\begin{aligned} D(ff_1)_{ik} &= 2I_i(ff_1)\,I_k(f_1f),\\ D(ff_1;\ f_1f_2)_{ik} &= 2I_i(ff_1)\,I_k(f_1f_2). \end{aligned} \tag{5.6}$$

The quantity $\Delta(ff_1)$ in Eq. (5.4) represents the energy required for the production of a single pair with electron and "hole" at the points f and f_1. In the expression (5.1) the terms containing $S_1(ff_1)$ are small quantities of the second order, and all the other terms are of the third order. The interaction between the electrons affects only the third order terms, since these are the only ones that contain the interaction integrals.

As was shown in Ref. 3, when the only external electric field is one stationary in time, the "deformed" current operator is non-vanishing only in third order. In our case part of the "deformed" current operator produced by the radiation field is non-vanishing already in the second approximation.

By applying the method of approximate diagonalization to the operator (5.1) one can find the average value of the electric current density. Then, comparing this with the macroscopic expression for the electric current density in the crystal in the field of the light wave,

$$\mathbf{j} = \frac{\varepsilon - 1}{4\pi}\frac{\omega^2}{c^2}\mathbf{A}(t) - \frac{\sigma}{c}\dot{\mathbf{A}}(t), \tag{5.7}$$

one can derive the dispersion formulas for the dielectric permittivity $\epsilon = \epsilon(\omega)$ and electric conductivity $\sigma = \sigma(\omega)$. Using the connection between these quantities and the optical constants, it is then easy to find the dispersion formulas for the index of refraction $n = n(\omega)$ and the extinction coefficient $k = k(\omega)$. If in these calculations we keep the third order terms, we have a derivation of the dispersion formulas with approximate inclusion of the effects of interaction between the electrons.

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*The summation is taken over all indices fσ that appear under summation signs in Eq. (5.1), except indices f appearing equal in pairs.

This opens up possibilities for the development of a many-electron theory of the optical properties of crystals within the framework of the quasi-homopolar approximation.

The practical realization of this program encounters mathematical difficulties in connection with the diagonalization of the operator (5.1).

Using Tiablikov's method of approximate diagonalization,[2,3] which is correct for weak perturbations of the system, it is possible in principle to carry out the indicated program for some particular case, for example for a ferromagnetic crystal in a state close to saturation.

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CONTRIBUTION TO THE THEORY OF ANTIFERROMAGNETISM AT LOW TEMPERATURES

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The energy spectrum of a uniaxial antiferromagnetic substance is obtained without the assumption of nominal magnetization of the spin sublattices in the ground state.

THE existing theory of antiferromagnetism[1] is based on the hypothesis (first proposed by Landau[2]) that an antiferromagnetic substance can be described in terms of two or more magnetic moments, which compensate each other in the absence of magnetic field. The construction of the microscopic theory of antiferromagnetism ordinarily begins with the Heisenberg model of exchange interaction and the assumption of a regular ordering of "left" and "right" spins in the ground state.[3,4]

As has been remarked by Landau, such an assumption is in contradiction with quantum mechanics: the spin component of an individual atom in a prescribed direction cannot have a definite value, because of the exchange interaction. On the other hand it appears that the experimental data do not contradict the results obtained by the use of this model.

The purpose of the present paper is to show that the energy spectrum of an antiferromagnetic substance, and thus all of its thermodynamic functions, can be obtained in a phenomenological way just from the assumption of two (or more) sublattices, without postulating nominal magnetization of each of the sublattices in the ground state.

1. The assumption of the existence of two (or several) sublattices can be reduced to the assumption that the state of an antiferromagnetic substance is characterized by the specification of two (or several) magnetic moments $\mathbf{M}_i(\mathbf{r})$ at each point. Here one picks out as the ground state the state with homogeneous values of each of the magnetic

moments, M_{i0}, such that $\sum M_{i0} = 0$. Such a description is clearly legitimate at low temperatures, for which only long-wave vibrations of the spin system are excited and the atoms of an elementary cell can be regarded as situated at a single point.

In what follows we shall consider for simplicity an antiferromagnetic substance consisting of two sublattices with magnetic moments $\mathbf{M}_1(\mathbf{r})$ and $\mathbf{M}_2(\mathbf{r})$. This treatment is surely applicable to uniaxial antiferromagnetic substances.

We write the Hamiltonian of the system in the following form:

$$\mathscr{H} = \int dV \left\{\alpha \mathbf{M}_1\mathbf{M}_2 - \mathbf{M}_1\mathbf{H} - \mathbf{M}_2\mathbf{H}\right.$$

$$+ \frac{1}{2}\lambda\left(M_{1x}^2 + M_{1y}^2 + M_{2x}^2 + M_{2y}^2\right) \tag{1}$$

where α, β, β_{12} are constants connected with the exchange interaction, and the last three terms describe the exchange interaction caused by the inhomogeneity of the spin system. As will be seen below, in antiferromagnetic substances $\beta - \beta_{12} > 0$. Furthermore it can be shown that

$$\beta,\ \beta_{12} \sim \alpha a^2 \sim \Theta_C / M^2 a,$$

where Θ_C is the Curie temperature and a is the lattice constant; if $M \sim \mu/a^3$ (μ is of the order of magnitude of the Bohr magnetron), then β, $\beta_{12} \sim a^2\Theta_C/\mu M$ and $\alpha \sim \Theta_C/\mu M \gg 1$. In the general case the term involving the spatial inhomogeneity of the magnetic moments is of the form

$$\beta_{iklm}^{ss'}\left(\partial M_{si} / \partial x_k\right)\left(\partial M_{s'l} / \partial x_m\right).$$

For a uniaxial substance, in virtue of the isotropic nature of the exchange interaction, we get the expression that appears in the Hamiltonian (1).

In the above, λ is the constant of the magnetic anisotropy (the z axis is taken along the preferred direction). We note that $\lambda \ll \alpha$, since the magnetic anisotropy is due to relativistic interactions: $\lambda \sim \alpha\,(v\ \ c)^2$, where v is the speed of the electrons and c is that of light. We have omitted from the Hamiltonian terms proportional to the squares of the magnetic moments, M_1^2 and M_2^2, since they do not make any contribution to the equations of motion of the magnetic moments (see below).

The magnetic field $\mathbf{H}$ is composed of the constant homogeneous external field H_0 and the magnetic field $\mathbf{h}$ of the spin waves, which satisfies the equations of magnetostatics,

$$\operatorname{curl} \mathbf{h} = 0, \quad \operatorname{div} \mathbf{h} = -4\pi \operatorname{div}(\mathbf{M}_1 + \mathbf{M}_2). \tag{2}$$

Starting with the Hamiltonian (1), we write the equations of motion of the magnetic moments:

$$\partial \mathbf{M}_s / \partial t = g\left[\mathbf{M}_s \times \mathbf{H}_{\mathrm{eff}}^s\right]; \quad \mathbf{H}_{\mathrm{eff}}^s = -\delta\mathscr{H} / \delta\mathbf{M}_s, \quad (s = 1,\ 2), \tag{3}$$

where $\delta\mathscr{H}/\delta M_s$ is the variational derivative of the Hamiltonian with respect to the magnetic moment and g is the gyromagnetic ratio ($g\hbar = \mu$).

Using Eqs. (2) and (3) (in analogy with the procedure used by Herring and Kittel[5]), we can find the magnetic branch of the energy spectrum of the antiferromagnetic substance in the neighborhood of the ground state. To do this we linearize (2) and (3), substituting for $\mathbf{M}_1$ and $\mathbf{M}_2$ the expressions

$$\mathbf{M}_1 = \mathbf{M} + \boldsymbol{\mu}_1; \qquad \mathbf{M}_2 = -\mathbf{M} + \boldsymbol{\mu}_2. \tag{4}$$

Taking $\mathbf{H}_0$ along the z axis and assuming that all the variable quantities ($\boldsymbol{\mu}$ and $\mathbf{h}$) are proportional to $e^{-i(\omega t - \mathbf{kr})}$, we get:

$$\begin{aligned} &i\omega(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2) - g\left[\mathbf{H}_0 \times (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right] \\ &= g\left\{\lambda + (\beta - \beta_{12})k^2\right\}\left[\mathbf{M} \times (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\right], \\ &-i\omega(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) + g\left[\mathbf{H}_0 \times (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\right] \\ &= 2g\,\mathbf{M} \times \left\{(\alpha - \lambda - (\beta - \beta_{12})k^2)(\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right. \\ &\left. - (8\pi\mathbf{k} / k^2)\left[\mathbf{k} \times (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right]\right\} \end{aligned} \tag{5}$$

Since the constant α is very large, the second equation of (5) can be considerably simplified:

$$i\omega(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2) - g\left[\mathbf{H}_0 \times (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_2)\right] = -2g\alpha\left[\mathbf{M} \times (\boldsymbol{\mu}_1 + \boldsymbol{\mu}_2)\right]. \tag{6}$$

In fact, $\lambda \ll \alpha$ and

$$\beta k^2 \sim (\Theta_C / \mu M)(ak)^2 \sim \alpha(ak)^2 \lll \alpha,$$

since we are of course interested only in the long-wavelength vibrations ($ak \ll 1$).

Neglect of the last term in the second equation of (5) is equivalent to the neglect of the magnetic field of the spin wave. This means that in an antiferromagnetic substance the proper magnetic field of the spin waves never leads to a change of the dispersion law, as occurs in a ferromagnetic substance (cf. Refs. 5 and 6).

From Eqs. (5) and (6) we have

$$\omega_{1,2} = \sqrt{2g^2\alpha M^2\left[\lambda + (\beta - \beta_{12})k^2\right]} \pm gH_0; \tag{7}$$

and the energy $\epsilon = h\omega$ of the spin wave is given by

$$\varepsilon_{12} = \mu M\sqrt{2\alpha\left[\lambda + (\beta - \beta_{12})k^2\right]} \pm \mu H_0. \tag{8}$$

From (7) or (8) it can be seen that the difference $\beta - \beta_{12}$ must be positive, since in the opposite case the ground state would have a finite value of k, i.e., the ground state would be inhomogeneous. This fact enables us to introduce the following notation:*

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*Essentially this is a definition of the two quantities H_a and θ_C. The former is the magnetic field that would have to be applied along the axis of easiest magnetization to make the antiferromagnetic state thermodynamically unstable;[3] the latter is of the order of the Curie temperature.

$$\sqrt{2\alpha\lambda}\, M = H_a, \quad \sqrt{2\alpha(\beta - \beta_{12})}\, \mu M / a = \Theta_C. \tag{9}$$

We then have from Eq. (8)

$$\varepsilon_{1,2} = \sqrt{(\mu H_a)^2 + \Theta_C^2 (ak)^2} \pm \mu H_0 \quad (H_0 < H_a). \tag{10}$$

For $H_0 = 0$ there is a two-fold degeneracy:

$$\varepsilon_{1,2} = \sqrt{(\mu H_a)^2 + \Theta_C^2 (ak)^2}. \tag{11}$$

For $k = 0$ the energy $\epsilon_{1,2}$ of the spin wave is equal to μH_a, and for sufficiently large k it is proportional to k:

$$\varepsilon_{1,2} \approx \begin{cases} \mu H_a + {}^1/_2\, \Theta_C (\Theta_C / \mu H_a)(ak)^2, & ak \ll \mu H_a / \Theta_C, \\ \Theta_C (ak), & ak \gg \mu H_a / \Theta_C. \end{cases} \tag{12}$$

Using Eq. (11) for the energy of a spin wave, we find by the usual formulas of statistical physics the spin part of the specific heat of an antiferromagnetic substance:

$$C_s = \begin{cases} (1/\pi\sqrt{2\pi})\, R (\mu H_a / \Theta_C)^3 (\mu H_a / T)^{1/2} e^{-\mu H_a / T} & (T \ll \mu H_a), \\ (4\pi^2 / 15)\, R (T / \Theta_C)^3 & (\Theta_C \gg T \gg \mu H_a). \end{cases} \tag{13}$$

2. For the determination of the temperature dependence of the magnetic susceptibility ($\chi_{\|}$ and $\chi_{\perp}$) we must know in particular the energy spectrum of the antiferromagnetic substance in a magnetic field perpendicular to the axis of easiest magnetization.

If we take the x axis along the magnetic field, in the ground state the magnetic moments of the two sublattices have the following components ($\lambda \ll \alpha$):

$$M_{1x} = M_{2x} = H / 2\alpha; \quad M_{1y} = M_{2y} = 0; \quad M_{1z} = -M_{2z} = \sqrt{M^2 - M_{1x}^2}. \tag{14}$$

We note that here each of the moments is parallel to its own effective field. Moreover, the expressions (14) make clear the physical meaning of the constant α: $1\ \alpha = \chi_{\perp 0}$, where $\chi_{\perp 0}$ is the value of the perpendicular component of the magnetic susceptibility at temperature zero.

Using linearized equations of motion analogous to the preceding, we find the energy spectrum in this case ($\mathbf{H}$ perpendicular to the axis of easiest magnetization):

$$\omega_1 = gM\sqrt{2\alpha(\lambda + (\beta - \beta_{12})k^2)}, \quad \omega_2 = gM\sqrt{2\alpha[\lambda + (\beta - \beta_{12})k^2] + (H/M)^2} \tag{15}$$

or (in the previous notation)

$$\varepsilon_1 = \sqrt{(\mu H_a)^2 + \Theta_C^2 (ak)^2}, \qquad \varepsilon_2 = \sqrt{(\mu H_a)^2 + (\mu H)^2 + \Theta_C^2 (ak)^2}. \tag{16}$$

Knowledge of the dependence of the energy of the spin waves on the magnetic field [Eqs. (10) and (16)] enables us to find the dependence of the magnetic susceptibility on the temperature:

$$\chi_{ik} = -\partial^2 \Omega / \partial H_i \partial H_k \big|_{H=0},$$

where $\Omega = T \sum_k \ln(1 - e^{-\epsilon_k/T})$ is the thermodynamic potential of the "spin wave gas."

After some simple calculations we get

$$\chi_{\|} = \begin{cases} \dfrac{1}{\pi\sqrt{2\pi}} \dfrac{\mu^2}{a^3\Theta_C} \left(\dfrac{\mu H_a}{\Theta_C}\right)^{3/2} \left(\dfrac{T}{\Theta_C}\right)^{1/2} e^{-\mu H_a / T} & (T \ll \mu H_a), \\ \dfrac{\mu^2}{3a^3\Theta_C} \left(\dfrac{T}{\Theta_C}\right)^2 & (\Theta_C \gg T \gg \mu H_a); \end{cases} \tag{17}$$

$$\chi_{\perp} = \begin{cases} \dfrac{1}{\alpha} + \dfrac{1}{(2\pi)^{3/2}} \dfrac{\mu^2}{a^3\Theta_C} \left(\dfrac{\mu H_a}{\Theta_C}\right)^{1/2} \left(\dfrac{T}{\Theta_C}\right)^{3/2} e^{-\mu H_a / T} & (T \ll \mu H_a), \\ \dfrac{1}{\alpha} + \dfrac{\mu^2}{12a^3\Theta_C} \left(\dfrac{T}{\Theta_C}\right)^2 & (\Theta_C \gg T \gg \mu H_a). \end{cases} \tag{18}$$

The temperature dependences found here for the specific heat and the magnetic susceptibility naturally agree with the results previously obtained by Néel by microscopic considerations.[1] Here they bring us to our goal of relating the phenomenological constants Θ_C and H_a, which we have introduced to the measurable quantities c_S, $\chi_{\|}$, and $\chi_{\perp}$.

3. One can, of course, deal in a similar way with more complicated magnetic structures, in particular ferrites, in which the magnetic moments of the sublattices do not compensate each other.

If there are no special reasons for supposing that the difference between the magnetic moments of the sublattices is small (as, for example, in the case of Fe_2O_3, according to Refs. 7 and 8), then on carrying out calculations similar to those above we easily find that the magnetic part of the spectrum consists of two branches (if there are two sublattices). One of these has a large activation energy, of the order of Θ_C and is of course not excited at low temperatures (this is the analog of the optical branch in the vibrations of compound lattices). The other one is analogous to the ordinary Bloch spin waves. For the case of two moments and at not too low temperatures it has the form (supposing $M_1 > M_2$):

$$\omega = \frac{2gM_1M_2}{M_1 - M_2}(\beta - \beta_{12})k^2. \qquad (19)$$

Therefore the magnetic part of the specific heat must be proportional to $T^{3/2}$, as it is for ordinary ferromagnetic substances.

In conclusion the writers take this occasion to thank L. D. Landau and I. M. Lifshitz for valuable discussions on the questions considered here.

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SOVIET PHYSICS JETP VOLUME 34(7), NUMBER 1 JULY, 1958

SCATTERING OF DIRAC PARTICLES BY A SHORT-RANGE CENTER OF FORCE WITH DAMPING TAKEN INTO ACCOUNT

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The theory of radiation damping is used to investigate elastic scattering of Dirac particles by a stationary short-range center of force. An equation is obtained for the scattering cross section A relation [Eq. (5)] is established between the scattering phase shifts predicted by the theory of radiation damping for Dirac particles and relativistic spinless particles.

IN the present work the theory of radiation damping is used to study elastic scattering of Dirac particles by an arbitrary short-range center of force. This has already been done[1] only for spinless particles (hereinafter we shall use the notation developed in that article and denote it by SK). In the present case we shall divide the wave functions into two groups according to the z component of the spin ($m_s = \pm\frac{1}{2}$), rather than according to the component of the spin in the direction of motion, as was done in SK. Then the fundamental integral equation [see Eq. (21) of SK] of the theory of radiation damping for elastic scattering of spin-$\frac{1}{2}$ particles becomes

$$(\varepsilon_n(l) - 1) H^{(n)}_{\mathbf{k}'\mathbf{k}}(l, m_{s'}, m_s) = \frac{kK}{8\pi^2 c\hbar i} \sum_{l', n', m_{s''}} \varepsilon_{n'}(l') \oint d\Omega'' H^{(n)}_{\mathbf{k}'\mathbf{k}''}(l, m_{s'}, m_{s''}) H^{(n')}_{\mathbf{k}''\mathbf{k}}(l', m_{s''}, m_s), \qquad (1)$$

where $\hbar\mathbf{k}$, $\hbar\mathbf{k}'$, and $\hbar\mathbf{k}''$ are the momenta of the particles in the initial, final, and intermediate states, respectively. The matrix element of the transition from the state $\mathbf{k}''$, $m_{s''}$ to the state $\mathbf{k}'$, $m_{s'}$ is given by

$$,H_{\mathbf{k}'\mathbf{k}''}(m_{s'}, m_{s''}) = \sum_{l,\,n} H^{(n)}_{\mathbf{k}'\mathbf{k}''}(l, m_{s'}, m_{s''}) = \sum_{l=0}^{\infty} \frac{4\pi b_l}{2l+1} \sum_{m=-l}^{l} Y_l^m(\mathbf{k}')\, Y_l^{m*}(\mathbf{k}'')\, b'^{+}(m_{s'})\, b''(m_{s''}), \tag{2}$$

where $Y_\ell^m(\mathbf{k}')$ is a spherical function normalized to unity which depends on the spherical coordinate angles θ' and φ' of the vector $\mathbf{k}'$, b_ℓ is the amplitude in the expansion of the external potential, and n and $n' = 1$ and 2 take account of the fact that for a given ℓ, the value of j may be $\ell + \frac{1}{2}$ (when $n = 1$) or $\ell - \frac{1}{2}$ (when $n = 2$).

The spin part of the transition matrix element is

$$b'^{+}(m_{s'})\, b''(m_{s''}) = \begin{cases} \frac{1}{2}\left(1+\frac{k_0}{K}\right) + \frac{1}{2}\left(1-\frac{k_0}{K}\right)[\cos\theta'\cos\theta'' + \sin\theta'\sin\theta''\exp\{2i(\varphi''-\varphi')m_{s'}\}] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{when } m_{s''} = m_{s'} \\ m_{s'}\left(1-\frac{k_0}{K}\right)[\cos\theta'\sin\theta''\, e^{-2i\varphi'' m_{s'}} - \sin\theta'\cos\theta''\exp\{-2i\varphi' m_{s'}\}] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{when } m_{s''} = -m_{s'}. \end{cases} \tag{3}$$

In what follows we shall use recursion relations which relate $\cos\theta\, Y_\ell^m(\mathbf{k})$ and $\sin\theta\, e^{\pm i\varphi}\, Y_\ell^m(\mathbf{k})$ with the functions $Y_{\ell\pm1}^{m\pm1}(\mathbf{k})$. This makes it possible to eliminate $\cos\theta$ and $\sin\theta\, e^{\pm i\varphi}$ from the matrix elements. After some mathematical manipulation we obtain the following expression for the components of matrix element of Eq. (2):

$$H^{(n)}_{\mathbf{k}'\mathbf{k}''}(l, m_{s'}, m_{s''}) = \frac{8\pi^2 c\hbar}{kK}\, c_{nl} \sum_{m=-(j+1/2)}^{j+1/2} \Omega^{(n)}_{lm}(\mathbf{k}', m_{s'})\, \Omega^{*(n)}_{lm}(\mathbf{k}'', m_{s''}), \tag{4}$$

where ($n = 1$ or 2)

$$c_{1l} = {}^1/_2(1 + k_0/K)\, c_l + {}^1/_2(1 - k_0/K)\, c_{l+1}, \quad c_{2l} = {}^1/_2(1 + k_0/K)\, c_l + {}^1/_2(1 - k_0/K)\, c_{l-1}, \tag{5}$$

and c_ℓ gives the scattering phase shift of a spinless particle. According to Eq. (52), of SK, the latter is related to the potential $V(r)$ by

$$c_l = \frac{kK}{2\pi c\hbar}\, \frac{b_l}{2l+1} = \frac{\pi K}{c\hbar} \int_0^{\infty} rV(r)\, J^2_{l+1/2}(kr)\, dr. \tag{6}$$

If we now set the components of the spherical spinor $\Omega^{(n)}_{\ell m}$ in Eq. (4) corresponding to the two values of the z component of the spin ($m = \pm\frac{1}{2}$) equal to

$$\Omega^{(n)}_{lm}(\mathbf{k}, m_s) = A^{(n),\, m_s}_{lm}\, Y_l^{m-m_s-1/2}(\mathbf{k}), \qquad A^{(1),\,1/2}_{lm} = A^{(2),\,-1/2}_{lm} = \sqrt{\frac{l+m}{2l+1}}, \quad -A^{(1),\,-1/2}_{lm} = A^{(2),\,1/2}_{lm} = \sqrt{\frac{l-m+1}{2l+1}},$$

we obtain the orthogonality of the components of the matrix elements, which is necessary in the theory of damping, namely

$$\sum_{m_{s''}} \oint d\Omega''\, H^{(n)}_{\mathbf{k}'\mathbf{k}''}(l, m_{s'}, m_{s''})\, H^{(n')}_{\mathbf{k}''\mathbf{k}}(l', m_{s''}, m_s) = \frac{8\pi^2 c\hbar}{kK}\, c_{nl}\, \delta_{nn'}\delta_{ll'}\, H^{(n)}_{\mathbf{k}'\mathbf{k}}(l', m_{s'}, m_s). \tag{7}$$

From (7) and (1) we obtain the following expression for the transition amplitudes $C'(m_{s'}) \equiv C(\mathbf{k}', m_{s'}, t)$ (see Eq. (15) of SK):

$$C'(m_{s'}) = \frac{1 - e^{ic(K'-K)t}}{c\hbar L^3 (K' - K)} \sum_{l,\,n} \varepsilon_n(l)\, H^{(n)}_{\mathbf{k}'\mathbf{k}}(l, m_{s'}, m_s),$$

where

$$\varepsilon_n(l) = 1/(1 + ic_{nl}),$$

and we simply do not take into account the coefficients β_1 and β_2, which are of order L^{-3}. It is necessary to include β_1 and β_2 in order to prove that

$$C^+(m_s)\,C(m_s) + \sum_{\mathbf{k}',\, m_{s'}}{}' C'^+(m_{s'})\,C'(m_{s'}) = 1.$$

The differential cross section for elastic scattering is

$$d\sigma_{m_{s'},\,m_s} = \frac{L^3K}{ck}\sum_{\mathbf{k}'}\frac{\partial}{\partial t}C'^+(m_{s'})\,C'(m_{s'}) = |\,f(m_s, m_{s'}, \theta', \varphi')\,|^2\,d\Omega'. \tag{8}$$

If the incident Dirac particle is directed along the z axis ($\cos\theta = 1$), the scattering amplitude is given by

$$f(m_s, m_{s'}, \theta', \varphi') = \begin{cases} -\dfrac{1}{k}\displaystyle\sum_{l=0}^{\infty}\left[(l+1)\frac{c_{1l}}{1+ic_{1l}} + l\,\frac{c_{2l}}{1+ic_{2l}}\right]P_l(\cos\theta') \text{ when } m_{s'} = m_s, \\ -\dfrac{1}{k}\displaystyle\sum_{l=0}^{\infty}\left[\frac{c_{1l}}{1+ic_{1l}} - \frac{c_{2l}}{1+ic_{2l}}\right]2m_{s'}e^{-2i\varphi' m_{s'}}\,P_l^1(\cos\theta') \\ \qquad\qquad\qquad \text{when } m_{s'} = -m_s. \end{cases} \tag{9}$$

The total effective elastic scattering cross section is

$$\sigma = \frac{1}{2}\sum_{m_{s'},\,m_s}\oint |\,f(m_s, m_{s'}, \theta', \varphi')\,|^2\,d\Omega' = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}\left[(l+1)\frac{c_{1l}^2}{1+c_{1l}^2} + l\,\frac{c_{2l}^2}{1+c_{2l}^2}\right] = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}\left[(l+1)\sin^2\eta_l^{(1)} + l\sin^2\eta_l^{(2)}\right]. \tag{10}$$

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ON THE DESTRUCTION AND THE ONSET OF SUPERCONDUCTIVITY IN A MAGNETIC FIELD

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Transition from the superconducting state to the normal and vice versa in the presence of an external magnetic field is considered. Critical magnetic field strengths H_c, H_{c1} and H_{c2}, which correspond to equilibrium transition and to the boundaries of the supercooled and superheated regions respectively, are computed. Cases of small samples and of bulk metals are considered.

THE destruction and the onset of superconductivity in the presence of an external magnetic field proceed in entirely different ways, depending on the dimensions, shape and internal condition of the sample (its purity, its homogeneity, etc.). In the simplest case of a bulk sample of cylindrical shape subjected to a field parallel to the axis of the cylinder, assuming that no intermediate state occurs, the equilibrium transition from the superconducting to the normal state (and vice versa) takes place in a field given by

$$H_{cb} = \sqrt{8\pi(F_{n0} - F_{s0})},$$

Here F_{n0} and F_{s0} are the free energy densities of the normal and superconducting phases. (The change of volume associated with the transition is neglected here and throughout the paper.)

Experimentally, however, supercooling of the normal phase can occur and in fact, is often observed, so that its transition to the superconducting phase occurs in fields $H_1 < H_{cb}$. Similarly, it is also found that the superconducting phase can experience a superheating, in which case it returns to the normal phase only for $H_2 > H_{cb}$. The values of H_1 and H_2 can not be calculated. They depend on many factors and, generally speaking, differ from experiment to experiment. Nevertheless, as was shown in previous work,[1,2] there exist certain critical fields, H_{c1} and H_{c2}, which serve as boundaries of the supercooling and superheating regions, such that $H_1 \geq H_{c1}$ and $H_2 \leq H_{c2}$ always. Thus the quantities H_{c1} and H_{c2} determine the maximum extent of the hysteresis loop, and their calculation is obviously of interest, especially since in certain cases H_{c1} has been reached experimentally.[2]

In small samples, i.e., when the smallest characteristic dimension of the sample, L, is commensurate with the penetration depth of a weak magnetic field into the metal (we shall denote this penetration depth by δ_0), the supercooling and superheating, generally speaking, are even more strongly pronounced than in the bulk metal. This is apparently due to the difficulty of producing sufficiently small nuclei. Indeed, in accordance with Ref. 1, the boundary between the normal and superconducting phases in bulk becomes blurred and has a width of the order of δ_0/κ. For pure superconductors, $\kappa \sim 0.1$ and, thus, it should be difficult to form nuclei in samples of dimensions on the order of $\delta_0/\kappa \sim 10\delta_0 \sim 10^{-4}$ cm.*

The circumstances noted above also hinder the formation of an intermediate state in small samples, to say nothing of the fact, that in this case, because of the increase of the surface energy density, stratification of the sample would become less advantageous even if the boundary between the phases were sharp. Because of this, one naturally expects more pronounced hysteresis in samples of small dimensions, and in such samples the attainment of the corresponding critical values H_{c1} and H_{c2} should be relatively easy. On the other hand, for very small samples, smaller than a certain dimension, $L_c \sim \delta_0$, which depends on the shape of the sample, the transition from the superconducting state to the normal state (and vice versa) is a second order transition[1] and, consequently, superheating and supercooling are impossible (in other words, in this case $H_{c1} = H_{c2} = H_c$, where H_c is the critical field for the equilibrium transition). Hence, it is clear that supercooling and superheating are pronounced only in some "average" domain of sample sizes.

Various aspects of this problem have already been considered in the literature.[1-7] However, because of the publication of some new results[8] of an investigation of the destruction of superconductivity in cadmium, this series of questions has again attracted attention. The corresponding results, which relate to the values of H_{c1}, H_{c2} and H_c in different cases, will be presented below.

1. Let us first write down the general thermodynamic relations for superconductors. The free energy density of a superconductor, according to Ref. 1, is given by

$$F_{sH} = F_{s0} + \frac{H^2}{8\pi} + \frac{1}{2M}\left|-i\hbar\nabla\Psi - \frac{e}{c}\mathbf{A}\Psi\right|^2, \qquad (1.1)$$

$$F_{s0} = F_{n0} + \alpha|\Psi|^2 + \frac{\beta}{2}|\Psi|^4, \qquad (1.2)$$

where $F_{n0}(T)$ is the free energy in the normal state in the absence of the magnetic field H; the function Ψ plays the role of the parameter η, which enters into the theory of second-order phase transitions, and α, β, and M are certain coefficients. It is possible, in principle, to use here an expression different from Eq. (1.2) to represent $F_{s0}(|\Psi|^2)$, but for definiteness, we shall use only the expression shown above, bearing in mind certain statements made in Refs. 9 and 10.

From the condition that the total free energy must be a minimum, these equations result:

$$\frac{1}{2M}\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2\Psi + \alpha\Psi + \beta|\Psi|^2\Psi = 0; \qquad (1.3)$$

$$-\operatorname{curl}\operatorname{curl}\mathbf{A} = \Delta\mathbf{A} = -\frac{4\pi}{c}\mathbf{j}_s,$$

$$\mathbf{j}_s = -\frac{ie\hbar}{2M}(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*) - \frac{e^2}{Mc}\mathbf{A}|\Psi|^2, \qquad (1.4)$$

where it is assumed that $\operatorname{div}\mathbf{A} = 0$.

In considering a superconductor in a uniform external field, H_0, the role of the thermodynamic potential which has a minimum at equilibrium is not played by $\int F_{sH} dV$, but rather by another quantity, which we shall call $\Phi(T, H_0)$. This difference, which is immaterial from the standpoint of deriving Eqs. (1.3) and (1.4), is related to the existence of an "energy of interaction" between the magnetic moment of the superconductor and the external magnetic field. The expression for Φ can be obtained by means of an analogy with the theory of magnetic materials, where that part of Φ which is a function of H_0, is determined by the expression $-\int \mathbf{M}\, d\mathbf{H}_0 dV$ (here $\mathbf{M}$ is the magnetization). If $M = \text{const} \times H$, then it follows that

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*Here we assume (as is usually done) that far from the critical temperature, T_c, $\delta_0 \sim 10^{-5}$ cm (near T_c the values of δ_0 and of $\delta_0/\varkappa$ are still larger). For samples with dimensions $L < \delta_0/\varkappa$, the formation of a transition region between phases is still possible in principle, but involves an increase in the corresponding surface energy.

$$\Phi_s(T, H_0) = \int \left\{F_{s0} + \frac{\hbar^2}{2M}(\nabla\Psi)^2\right\} dV - \frac{\beta \mathbf{H}_0}{2}, \qquad (1.5)$$

where $\mu = \int \mathbf{M}\, dV$ is the magnetic moment of the solid.

The derivation given above is not sufficiently rigorous, but we also arrive at Eq. (1.5) by taking it into account that during an isothermal process at equilibrium in a magnetic field, the following quantity must have a minimum:[5]

$$\Phi(T, H_0) = E - TS - \frac{1}{4\pi}\int \mathbf{B}\mathbf{H}_0 dV + \frac{1}{8\pi}\int H_0^2 dV, \qquad (1.6)$$

where E is the internal energy of the solid, S is its entropy, and the term containing H_0^2 is added for convenience. In the case of a superconductor:

$$E - TS = \int F dV = \int F_{sH} dV,$$

where F_{sH} is defined by Eq. (1.1).

In most cases the function Ψ can be considered real. Indeed, in transforming to gradients, it is necessary that

$$\Psi' = \Psi \exp\left[i\frac{e}{\hbar c}\chi(\mathbf{r})\right], \quad \mathbf{A}' = \mathbf{A} + \nabla\chi,$$

where the only requirements on $\chi(\mathbf{r})$ are that the function Ψ' remain everywhere continuous and single valued. For a simply connected superconductor, this condition can be satisfied everywhere and at the same time one can choose χ so that the function Ψ' turns out to be real.

In this case, as is evident from Eq. (1.4), if $\Psi = \Psi^*$ (the primes on Ψ are omitted),

$$\mathbf{j}_s = -(e^2/Mc)\Psi^2\mathbf{A}. \qquad (1.7)$$

For a bulk, multiply-connected superconductor it is possible to choose Ψ real only if the magnetic flux through any contour located in the interior of the superconductor is equal to zero. Indeed, in the interior of the superconductor the current density is $j_s = 0$, but on the other hand, under condition (1.7), the circulation of the current on such a contour is given by

$$\oint \mathbf{j}_s ds = \text{const} \cdot \oint \mathbf{A}_s ds = \text{const} \cdot \int H_n d\sigma,$$

i.e., it is proportional to the magnetic flux through the contour. Therefore, if this flux is different from zero, Eq. (1.7) cannot be true. The magnetic flux through a superconductor does not vanish, as is well known, only for multiply-connected samples which are transformed into the superconducting state in the presence of a magnetic field. This case will not be considered below, and Eq. (1.7) will be used when we have for the superconductor [see Eq. (1.6)]

$$\Phi_s(T, H_0) = \int [F_{s0} + \frac{\hbar^2}{2M}(\nabla\Psi)^2] dV - \frac{1}{2c}\int \mathbf{j}_s\mathbf{A} dV + \frac{1}{8\pi}\int (\mathbf{H} - \mathbf{H}_0)^2 dV, \qquad (1.8)$$

where, as before, the integration is performed over all space. (Outside the superconductor, naturally, $\Psi = 0$ and $j_s = 0$; in Eq. (1.6), in addition, the substitution $\mathbf{B} = \mathbf{H}$ was made since the metal is considered to be non-magnetic.)

In the normal state, since the metal is non-magnetic,

$$\Phi_n(T, H_0) = \int F_{n0} dV. \qquad (1.9)$$

Eq. (1.8) can be transformed into the form of Eq. (1.5) by using the field equations.

The function Ψ which enters into Φ_s is determined in the equilibrium or metastable state by the condition that Φ_s must have a minimum. As has been pointed out, this condition is the same as the requirement of a minimum in $\int F_{sH} dV = E - TS$, i.e., it is determined by means of Eq. (1.3). The equivalence of the expression for Φ_s and $E - TS$ in this respect is evident from Eq. (1.6), since the presence of the term

$$-\frac{\mathbf{H}_0}{4\pi}\int \mathbf{H}\, dV = -\frac{\mathbf{H}_0}{4\pi}\int \text{curl}\, \mathbf{A}\, dV$$

is immaterial when Ψ and $\mathbf{A}$ are varied for the purpose of obtaining Eqs. (1.3) and (1.4). In this case, in the absence of a field, when one can write $\mathbf{A} = 0$ and $\nabla\Psi = 0$, we have

$$\Psi^2 = \Psi_\infty^2 = -\frac{\alpha}{\beta} = \frac{|\alpha|}{\beta}, \quad F_{s0}(\Psi_\infty^2) = F_{n0} - \frac{\alpha^2}{2\beta}. \qquad (1.10)$$

For a bulky cylindrical superconductor in a field, parallel to the axis of the cylinder, one can assume the field to be H_0 outside the metal and by $H = 0$ in the interior of the sample. Therefore

$$\mu = \frac{1}{4\pi}\int (\mathbf{H}-\mathbf{H}_0)\,dV = -\frac{\mathbf{H}_0 V}{4\pi},$$

where V is the volume of the cylinder. The term with $(\nabla\Psi)^2$ in Eq. (1.5), which differs from zero only near the surface of the metal,* can be neglected in our case and, in this way, in the equilibrium state, when the external field H_0 is equal to the critical field H_{cb} for bulk metal, we have

$$H_{cb}^2/8\pi = F_{n0} - F_{s0} = \alpha^2/2\beta. \qquad (1.11)$$

Let us introduce the notation

$$\Psi_0^2 = \frac{\Psi^2}{\Psi_\infty^2}, \quad \delta_0^2 = \frac{Mc^2}{4\pi e^2\Psi_\infty^2} = \frac{mc^2}{4\pi e^2\Psi_\infty^2} = \frac{2.84\cdot 10^{11}}{\Psi_\infty^2}\ \text{cm}^2,$$

$$\varkappa = \frac{Mc}{e\hbar}\sqrt{\frac{\beta}{2\pi}} = \frac{\sqrt{2}\,e}{\hbar c}H_{cb}\delta_0^2 = 2.17\cdot 10^7 H_{cb}\delta_0^2. \qquad (1.12)$$

Here we have taken it into account that δ_0 (the depth of penetration of a weak magnetic field into a superconductor) is the observable quantity, while the "concentration of superconducting electrons", $n_s = \Psi_\infty^2$ manifests itself through δ_0^2. Therefore, in the expression for δ_0^2, the coefficient M, which plays the role of an effective mass, can be set equal to the mass of the free electron m, thus uniquely relating Ψ_∞^2 with δ_0^2. As for the charge e, it is assumed above to equal the free electron charge, an assumption having rather firm theoretical foundations.[1,11] Making use of Eqs. (1.12) and (1.9), the potential (1.5) can be written in the form:

$$\Phi_s = \Phi_n + \frac{H_{cb}^2}{8\pi}\int\left[\Psi_0^4 - 2\Psi_0^2 + \frac{2\delta_0^2}{\varkappa^2}(\nabla\Psi_0)^2\right]dV - \frac{1}{2}\mu H_0. \qquad (1.13)$$

From (1.13) it is clear that only two parameters, H_{cb} and δ_0, enter into the theory [the parameter κ can be expressed in terms of these other two; see Eq. (1.12)], so that only the penetration depth δ_0 enters into the gradient term.

In terms of these same symbols and for a real Ψ, Eqs. (1.3) and (1.4) can be rewritten

$$\nabla^2\Psi_0 = \frac{\varkappa^2}{\delta_0^2}\left[\Psi_0^3 - \Psi_0 + \frac{1}{2\delta_0^2 H_{cb}^2}\Psi_0 A^2\right], \qquad (1.14)$$

$$\text{curl curl}\,\mathbf{A} + \delta_0^{-2}\Psi_0^2\mathbf{A} = 0.$$

Note, finally, that the temperature dependence of α and β, or, equivalently, of the quantities H_{cb} and δ_0, is not fixed, and can be arbitrary within wide limits. From experimental data, it is known that quite good agreement with experiments can be usually achieved by putting

$$H_{cb} = H_{00}[1-(T/T_c)^2], \quad \delta_0 = \delta_{00}[1-(T/T_c)^4]^{-1/2}. \qquad (1.15)$$

2. For a number of pure metals $\kappa \ll 1$ (e.g., for aluminum at $T \to 0$, according to Ref. 9, κ 0.05). Under such conditions, for certain problems, we can simply let $\kappa = 0$, which is the same as assuming the Ψ does not depend on the coordinates [see Eq. (1.14)]. One can proceed exactly in the same manner in analyzing the destruction of superconductivity in small samples, when

$$(\varkappa L/\delta_0)^2 \ll 1, \qquad (2.1)$$

where L is the characteristic dimension of the sample (the thickness of a film, the radius of a small sphere or a cylinder, etc.). Let us first dwell on this particular case.

According to Eq. (1.13) we have here

$$f = \frac{\Phi_s - \Phi_n}{(H_{cb}^2/8\pi)V} = \Psi_0^4 - 2\Psi_0^2 - \frac{4\pi}{H_{cb}^2 V}\mu H_0, \qquad (2.2)$$

where V is the volume of the sample.

Since in the superconducting state, whether in equilibrium or in a metastable state, the potential Φ_s (or the potential difference f) should have a minimum, the quantities H_0 and Ψ_0 are related by the equation:

$$\partial f(H_0, \Psi_0)/\partial\Psi_0 = 0, \qquad (2.3)$$

where for simplicity the value of Ψ_0, associated with the minimum, is represented by the same symbol, Ψ_0.

The equilibrium transition between the superconducting and normal phases occurs if

$$f(H_c, \Psi_c) = 0, \qquad (2.4)$$

where H_c and Ψ_c are corresponding values of H_0 and Ψ_0. The second equation needed to determine H_c and Ψ_c is clearly Eq. (2.3) with $H_0 = H_c$ and $\Psi_0 = \Psi_c$.

If the characteristic dimension L is less than a certain value L_c, then with increasing H_0 the function Ψ_0 decreases monotonically and $\Psi_c = 0$. In this region there occurs, as is evident, a second-order transition, when neither superheating or supercooling is possible. This is clearly seen in Fig. 1, in which we have plotted the potential difference f as a function of Ψ_0 for various values of $(H_0/H_{cb})^2$ for small spheres of radius $a = \delta_0$ [see Eqs. (2.19), (2.23) and (2.25) below].

The value $L = L_c$ corresponds to the critical Curie point, when the specific-heat discontinuity is infinite at the transition. For $L > L_c$, a first-order transition takes place. Plots of $f(\Psi_0)$ for this case are presented in Fig. 2 for small spheres with $a/\delta_0 = 8$. It is already evident from this fig-

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*It is assumed that all of the sample becomes superconducting.

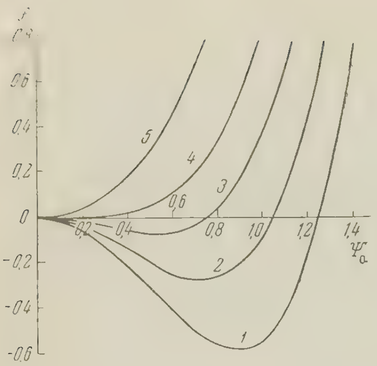

FIG. 1. Curves 1 through 5 correspond to values of $(H_0/H_{cb})^2$ equal to 5, 10, 15, 20, and 30, respectively.

ure, that the metastable (supercooled) normal phase, which accounts for the relative minimum in f at $\Psi_0 = 0$, can exist only in fields $H_0 > H_{C1}$. The value of H_{C1} is obtained from the condition

$$\partial^2 f(H_{c1},0)/\partial\Psi_0^2 \equiv (\partial^2 f/\partial\Psi_0^2)_{H_0=H_{c1},\Psi_0=0} = 0. \qquad (2.5)$$

As for the metastable (superheated) superconducting phase, it can exist only for fields $H_0 < H_{C2}$, where for $\Psi_{C2} \neq 0$:

$$\partial^2 f(H_{c2}, \Psi_{c2})/\partial\Psi_0^2 = 0, \ \partial f(H_{c2},\Psi_{c2})/\partial\Psi_0 = 0. \qquad (2.6)$$

In this case, when $\Psi_0 = \text{const.}$, the magnetic moment of the sample is calculated in the same way as in the Londons' theory [see the second of Eqs. (1.14)]. The solutions of the corresponding problems are presented in detail in Ref. 12. For a field parallel to a thin film of thickness 2d

$$\frac{\mu}{V} = \frac{\mu}{2dS} = -\left(1 - \frac{\tanh(\Psi_0 d/\delta_0)}{\Psi_0 d/\delta_0}\right)\frac{H_0}{4\pi}. \qquad (2.7)$$

For a sphere of radius a:

$$\frac{\mu}{V} = \frac{\mu}{4\pi a^3/3} = -3\left\{1 - \frac{3\delta_0}{\Psi_0 a}\coth\frac{\Psi_0 a}{\delta_0} + \frac{3\delta_0^2}{(\Psi_0 a)^2}\right\}\frac{H_0}{8\pi}. \qquad (2.8)$$

For a circular cylinder with radius r in a field parallel to its axis,

$$\frac{\mu}{V} = \frac{\mu}{\pi r^2 l} = -\left[1 - \frac{2\delta_0}{\Psi_0 r}\frac{I_1(\Psi_0 r/\delta_0)}{I_0(\Psi_0 r/\delta_0)}\right]\frac{H_0}{4\pi}$$
$$= -\frac{I_2(\Psi_0 r/\delta_0)}{I_0(\Psi_0 r/\delta_0)}\frac{H_0}{4\pi}, \qquad (2.9)$$

where $I_n(x) = i^{-n}J_n(ix)$ and J_n is the Bessel function. For the case of a circular cylinder perpendicular to the field, the value of μ/V is twice as large as that given in Eq. (2.9).

For a film, the quantity f assumes a form that is clear from Eqs. (2.2) and (2.7), and Eqs. (2.3) and (2.4) yield*

$$\left(\frac{H_0}{H_{cb}}\right)^2 = \frac{4\Psi_0^2(\Psi_0^2 - 1)\cosh^2(\Psi_0 d/\delta_0)}{1 - \sinh(2\Psi_0 d/\delta_0)/(2\Psi_0 d/\delta_0)}; \qquad (2.10)$$

$$\left(\frac{H_c}{H_{cb}}\right)^2 = \frac{\Psi_c^2(2 - \Psi_c^2)}{1 - \tanh(\Psi_c d/\delta_0)/(\Psi_c d/\delta_0)}. \qquad (2.11)$$

Using the first relation in (2.6) we find:

$$\left(\frac{H_{c2}}{H_{cb}}\right)^2 = \frac{2\Psi_{c2}^2(1 - 3\Psi_{c2}^2)\cosh^3(\Psi_{c2}d/\delta_0)}{\cosh\left(\frac{\Psi_{c2}d}{\delta_0}\right)\left[1 - \frac{\sinh(2\Psi_{c2}d/\delta_0)}{2\Psi_{c2}d/\delta_0}\right] + \frac{\Psi_{c2}d}{\delta_0}\sinh\left(\frac{\Psi_{c2}d}{\delta_0}\right)}. \qquad (2.12)$$

The second equation for H_C and Ψ_C or H_{C2} and Ψ_{C2} is obtained, as has already been remarked, from Eq. (2.10) by replacing H_0 and Ψ_0 with H_C and Ψ_C or with H_{C2} and Ψ_{C2}. The value of the field H_{C1} determined from Eq. (2.5) is equivalent to Eq. (2.12) with $\Psi_{C2} \to 0$. This same result is obtained directly from Eqs. (2.10) and (2.11) also for $\Psi_0 \to 0$, i.e., by expanding into a series in terms of $\Psi_0 d/\delta_0$. The result is:

$$H_{c1}/H_{cb} = \sqrt{6}\,\delta_0/d. \qquad (2.13)$$

For $d < d_C$, where d_C is the half thickness associated with the Curie point (in this case $L_C = 2d_C$), we have $H_{C1} = H$ i.e., Eq. (2.13) also determines the critical field in the region of a second-order transition.

From Eq. (2.10), for small Ψ_0, as a result of expanding the hyperbolic functions into a series and retaining terms of the order of $(\Psi_0 d/\delta_0)$,* we obtain

$$\Psi_0^2 \simeq \frac{1 - {}^1/_6(H_0/H_{cb})^2(d/\delta_0)^2}{1 - (d/\delta_0)^2 + {}^1/_{30}(H_0/H_{cb})^2(d/\delta_0)^4}$$
$$\sim \frac{1 - {}^1/_6(H_0/H_{cb})^2(d/\delta_0)^2}{1 - {}^4/_5(d/\delta_0)^2}. \qquad (2.14)$$

Hence for $\Psi_0 \to 0$, one necessarily obtains Eq. (2.13), but, in addition to this, it is clear that the form of the solution is changed for different values of d/δ_0. Thus, if $d/\delta_0 < \sqrt{5}/2$ then for $H_0 > H_{C1}$ no real solution for Ψ_0 exists in general. On the other hand, when $d/\delta_0 > \sqrt{5}/2$, such a solution does exist. This denotes the presence of a first-order transition. Thus, for a film

$$d_c/\delta_0 = \sqrt{5}/2 = 1.12. \qquad (2.15)$$

The character of the $\Psi_0(H_0)$ curves is shown in Fig. 3, as a function of different sample dimensions, for the case of small spheres (see below). For

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*The case of a thin film was previously considered in Ref. 3, where, however, the expression for the field H_{c2} was not included.

*The author wishes to express his thanks for performing the numerical calculations to T. I. Bachelis, E. I. Gusev, L. V. Pariiskaia and F. I Strizhevskii.

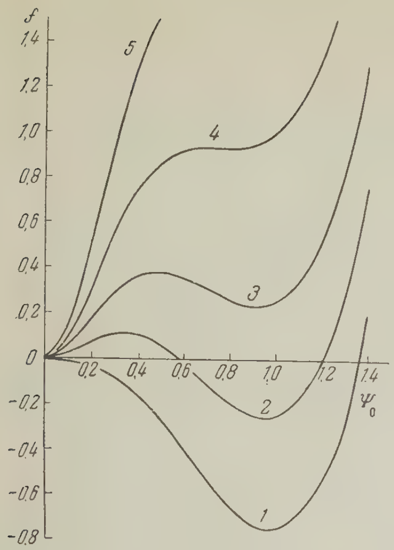

FIG. 2. Curves 1 through 5 correspond to values of $(H_0/H_{cb})^2$ equal to 0.25, 0.75, 1.25, 2.00, and 3.00, respectively.

sufficiently thick films, when

$$d/\delta_0 \gg 1 \tag{2.16}$$

and Ψ_0 is of the order of unity, Eqs. (2.11) and (2.12) are greatly simplified. They assume the form

$$\frac{H_c}{H_{cb}} = 1 + \frac{\delta_0}{2d}, \quad \Psi_c = 1 - \frac{\delta_0}{8d}; \tag{2.17}$$

$$\frac{H_{c2}}{H_{cb}} = \frac{2\sqrt{2}}{5}\left(\frac{27}{5}\right)^{1/4}\sqrt{\frac{d}{\delta_0}} = 0.86\sqrt{\frac{d}{\delta_0}}; \tag{2.18}$$

$$\Psi_{c2} = \sqrt{\frac{3}{5}} = 0.772.$$

Eq. (2.13) for H_{c1}/H_{cb} remains valid also when condition (2.16) is satisfied.

For small spheres we have $(x_i = \Psi_i a/\delta_0)$:

$$f = \Psi_0^2(\Psi_0^2 - 2) + \frac{3}{2}\left(\frac{H_0}{H_{cb}}\right)^2\left[1 - \frac{3\coth x_0}{x_0} + \frac{3}{x_c^2}\right]; \tag{2.19}$$

$$\left(\frac{H_0}{H_{cb}}\right)^2$$

$$= \frac{8}{9}\Psi_0^2(1 - \Psi_0^2)\sinh^2 x_0\left[1 + \frac{\sinh 2x_0}{2x_0} - \frac{2\sinh^2 x_0}{x_0^2}\right]^{-1}; \tag{2.20}$$

$$\left(\frac{H_c}{H_{cb}}\right)^2 = \frac{2}{3}\Psi_c^2(2 - \Psi_c^2)\left[1 - \frac{3\coth x_c}{x_c} + \frac{3}{x_c^2}\right]^{-1}; \tag{2.21}$$

$$\left(\frac{H_{c2}}{H_{cb}}\right)^2 = \frac{4(1 - 3\Psi_{c2}^2)\Psi_{c2}^2 x_{c2}^2 \sinh^2 x_{c2}}{9[3\sinh^2 x_{c2} - x_{c2}^3\coth x_{c2} - x_{c2}^2 - {}^1/_2 x_{c2}\sinh 2x_{c2}]}; \tag{2.22}$$

$$H_{c1}/H_{cb} = 2\sqrt{5}\,\delta_0/a. \tag{2.23}$$

For small Ψ_0, it follows from Eq. (2.21) that

$$\Psi_0^2 \approx \frac{1 - {}^1/_{20}(H_0/H_{cb})^2(a/\delta_0)^2}{1 - {}^4/_{21}(a/\delta_0)^2}, \tag{2.24}$$

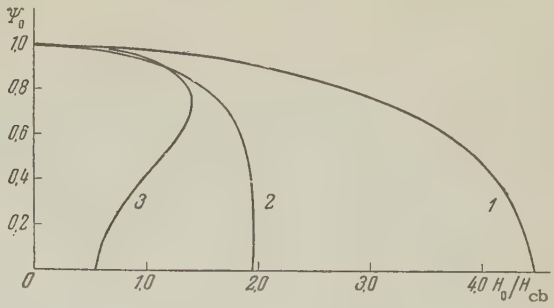

FIG. 3. Curve 1 – $a/\delta_0 = 1.00$; 2 – $a/\delta_0 = 2.29$; 3 – $a/\delta_0 = 8.00$.

whence we obtain Eq. (2.23), as well as the critical value of the radius, a_c, associated with the onset of first-order transitions (for $a > a_K$):

$$a_c/\delta_0 = \sqrt{21}/2 = 2.29. \tag{2.25}$$

For

$$a/\delta_0 \gg 1 \tag{2.26}$$

we obtain

$$H_c/H_{cb} = \sqrt{{}^2/_3}(1 + 3\delta_0/2a), \quad \Psi_c = 1 - 3\delta_0/8a; \tag{2.27}$$

$$H_{c2}/H_{cb} = {}^4/_5 \cdot 15^{-1/4}\sqrt{a/\delta_0} = 0.407\sqrt{a/\delta_0}, \quad \Psi_{c2} = \sqrt{{}^3/_5}. \tag{2.28}$$

The values of H_{c1}/H_{cb}, H_c/H_{cb} and H_{c2}/H_{cb}, obtained from Eqs. (2.20) through (2.23), are shown in Fig. 4. In exactly the same way as was done for the film and the small sphere, it is possible to determine the fields H_{c1}, H_c and H_{c2} for a cylinder by using Eqs. (2.2) through (2.9). It is especially easy to show by expanding the function I_n in Eq. (2.9) in a series, that for a cylinder, in a parallel field

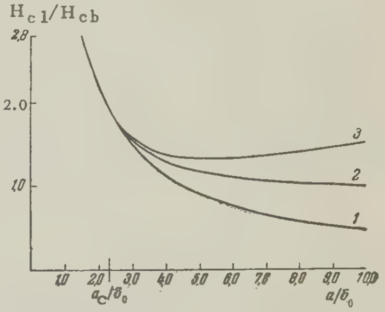

FIG. 4. Curve 1 – H_{c1}/H_{cb}; 2 – H_c/H_{cb}; 3 – H_{c2}/H_{cb}.

$$H_{c1}/H_{cb} = 4\delta_0/r, \quad r_c/\delta_0 = \sqrt{3}, \tag{2.29}$$

and for a cylinder, with a perpendicular field

$$H_{c1}/H_{cb} = \sqrt{8}\,\delta_0/r, \quad r_c/\delta_0 = \sqrt{3}. \tag{2.30}$$

Equations (2.20), (2.23), (2.25), (2.29) and (2.30) were previously obtained by Silin,[5] but in a somewhat more cumbersome way [this was because he did not make use of the general expression Eq. (1.13)]. The case of small spheres was also investigated in Ref. 4, but with the use of a different, more complicated, and at the same time, apparently less justifiable functional relation between F_{s0} and Ψ^2. The results obtained, concerning the dependence of H_{c1}, H_c and H_{c2} on the dimensions of the samples at least in their qualitative relations, are confirmed by experiments (see Refs. 13 through 16, 4, and 10).

3. If one makes use of the equations obtained in Sec. 2, one can conclude, that as L/δ_0 increases, the field H_{c1} approaches zero, and the field H_{c2} approaches infinity [see Eqs. (2.13) and (2.18) or (2.23) and (2.28)]. In actuality, however, all these expressions are strictly applicable only subject to condition (2.1), which does not permit consideration of sufficiently thick samples. By taking into account the parameter κ, the values of the fields H_{c1}, H_c and H_{c2} are changed, and, for example, for thin films, one obtains[7] to terms in κ^2

$$\left(\frac{H_{c1}}{H_{cb}}\right)^2 = 6\left(\frac{\delta_0}{d}\right)^2\left[1+\frac{8}{105}\left(\frac{\varkappa d}{\delta_0}\right)^2\right]. \tag{3.1}$$

Hence, it follows that Eq. (2.13) can be used in practice, as long as $\kappa d/\delta_0 \lesssim 1$ or, if $\kappa \sim 0.1$, as long as $d/\delta_0 \lesssim 10$. Since $\delta_0 \gtrsim 5 \times 10^{-6}$ cm, this means that the thickness of the film $2d$ can reach 10^{-4} cm or even somewhat more (except near T_c, where δ_0 increases.)

A general investigation of the critical field values in the region $\kappa L/\delta_0 \sim 1$ is rather complicated, and we shall therefore dwell only on the second limiting case, when

$$\varkappa L/\delta_0 \gg 1. \tag{3.2}$$

Since the width of the transition region between the superconducting and normal phases at equilibrium (i.e., for $H = H_{cb}$) is of the order of δ_0/κ (see Ref. 1.), condition (3.2) has an obvious physical meaning. Samples whose smallest dimensions L satisfy the inequality (3.2) will behave like bulk samples. For such samples, the boundary of the region of supercooling of the normal phase is determined from the condition of stability of this phase with respect to a transition into the superconducting state.[1,2] Thus

$$H_{c1}/H_{cb} = \sqrt{2}\varkappa. \tag{3.3}$$

In external fields such that $H_0 < H_{c1}$ the normal phase is unstable and cannot exist.

By comparing (3.3) with (2.13), (2.23), (2.29), and (2.30), which are valid in the second limiting case (2.1), the following becomes clear. As L increases, H_{c1} first decreases as $H_{c1}/H_{cb} = \text{const.} \times \delta_0/L$, but afterwards, when $\kappa L/\delta_0 \gg 1$, this decrease slows down and for the field H_{c1}, probably approaches monotonically the constant limit (3.3).

As regards the equilibrium critical field H_c, it approaches H_{cb} for cylindrical samples with axes parallel to the field, when condition (3.2) applies. For a slab of thickness $2d \gg \delta_0/\kappa$ we have[1,3]

$$H_c/H_{cb} = 1 + (\delta_0/2d)(1 + \varkappa/8\sqrt{2}). \tag{3.4}$$

The limit of the superheated region of the superconducting phase, i.e., the field H_{c2}, increases in certain ranges of values of L according to the law $H_{c2}/H_{cb} = \text{const}\ \sqrt{L/\delta_0}$ [see Eqs. (2.18) and (2.28)]. For $\kappa L/\delta_0 \sim 1$, this increase slows down and when condition (3.2) applies, the field H_{c2}, most probably, approaches smoothly a certain constant limit, which is a function of κ.

We now proceed to calculate H_{c2} for the bulk metal, considering for this purpose the superconducting half-space, on whose boundary (at $z = 0$) the external magnetic field, parallel to this boundary, is equal to H_0. The problem is one dimensional and Eqs. (1.14) become

$$d^2\Psi_0/d\xi^2 = \varkappa^2[-(1-a^2)\Psi_0 + \Psi_0^3], \tag{3.5}$$

$$d^2a/d\xi^2 = \Psi_0^2 a, \tag{3.6}$$

where

$$\begin{gathered}\xi = z/\delta_0, \quad a = A/\sqrt{2}\,H_{cb}\delta_0,\\ h = da/d\xi = H/\sqrt{2}\,H_{cb}.\end{gathered} \tag{3.7}$$

Equations (3.5) and (3.6) must be solved subject to the boundary conditions[1]

$$\begin{aligned}&\xi = \infty\colon\ \Psi_0^2 = 1,\ d\Psi_0/d\xi = 0,\ h = 0,\ a = 0,\\ &\xi = 0\colon\quad h = (da/d\xi)_0 = h_0,\ d\Psi_0/d\xi = 0,\end{aligned} \tag{3.8}$$

where $h_0 = H_0/\sqrt{2}\,H_{cb}$ is the field on the boundary in the new units, and the value $\xi = \infty$ corresponds to a layer of superconductor.

From Eqs. (3.5), (3.6), and (3.8), it follows that

$$(da/d\xi)^2 \equiv h^2 = {}^1/_2 - \frac{1}{\varkappa^2}\left(\frac{d\Psi_0}{d\xi}\right)^2 - (1-a^2)\Psi_0^2 + \Psi_0^4/2. \tag{3.9}$$

Making use of the integral of Eq. (3.9), and denoting $d\Psi_0/d\xi$ by y, we can write Eqs. (3.5) and (3.6) as

$$\begin{gathered}d\Psi_0/da = y\,[{}^1/_2 - (y/\varkappa)^2 - (1-a^2)\Psi_0^2 + \Psi_0^4/2]^{-1/2},\\ dy/da = \varkappa^2[-(1-a^2)\Psi_0 + \Psi_0^3]\,[{}^1/_2 - (y/\varkappa)^2 - (1-a^2)\Psi_0^2\\ + \Psi_0^4/2]^{-1/2}.\end{gathered} \tag{3.10}$$

The problem consists of finding an integration curve, which satisfies (3.10), goes out from the singular point $y = 0$, $a = 0$, $\Psi_0 = 1$, and intersects the plane $y = 0$ (in a coordinate system with variables y, a and Ψ_0) at the point a_0 and $\Psi_0(0)$, so that in accordance with (3.9)

$$h_0^2 = {}^1/_2 - (1 - a_0^2)\,\Psi_0^2(0) + \Psi_0^4(0)/2. \qquad (3.11)$$

For a given applied field, h_0, Eq. (3.11) determines a certain curve in the plane $y = 0$. Near the singular point one can write $\Psi = 1 + \varphi$, where $|\varphi| \ll 1$, and, thus, the equation can be made linear. Consequently, near the singular point,

$$\begin{aligned} \Psi_0 &= 1 + \varkappa^2 a^2/2\,(2 - \varkappa^2) + C\,(-a)^{\sqrt{2}\,\varkappa}, \\ y &= -\varkappa^2 a^2/(2 - \varkappa^2) - \sqrt{2}\,\varkappa C\,(-a)^{\sqrt{2}\,\varkappa}, \end{aligned} \qquad (3.12)$$

where C is an integration constant.

The behavior of the integral curves at sufficiently large values of $|a|$ is determined from Eq. (3.10) and, generally speaking, can be calculated quantitatively only by numerical methods. In the case of small κ, it turns out that it is more effective to get a direct solution of Eqs. (3.5) and (3.6), using, for large values of ξ, a solution equivalent to (3.12), obtained by the method of successive approximations.[1] As a result of such calculations, one can establish that the sought solution exists only so long as the field h_0 is less than a certain field $h_{c2} = H_{c2}/\sqrt{2}\,H_{cb}$, which plays the role of the boundary for superheating of the superconducting phase. Incidentally, the very fact that the field H_{c2} exists can be ascertained, naturally, even without numerical calculations. Thus, in the limiting case of very large κ, $H_{c2} = H_{cb}$. In fact, as $\kappa \to \infty$, we have from Eqs. (3.5) and (3.6)

$$\begin{aligned} \Psi_0^2 &= 1 - a^2, \quad a = \sqrt{2}/\cosh(\xi + C), \\ h &= da/d\xi = \sqrt{2}\,\sinh(\xi + C)/\cosh^2(\xi + C), \\ H/H_{cb} &= \sqrt{2}\,h = 2\sinh(\xi + C)/\cosh^2(\xi + C), \end{aligned} \qquad (3.13)$$

where C is an integration constant, and we have taken into account the boundary condition $a(\xi = \infty) = 0$. From the requirement, that $H = H_0$ when $\xi = 0$ (i.e., $z = 0$), the following condition is obtained:

$$\cosh^2 C = 2\,(H_{cb}/H_0)^2 \pm 2\sqrt{(H_{cb}/H_0)^4 - (H_{cb}/H_0)^2}.$$

Hence the possible existence of a solution only when $H_0 \leq H_{cb}$ is evident, since otherwise $\cosh^2 C$ becomes complex. Thus, when $\kappa \to \infty$, $H_{c2} = H_{cb}$. For $\kappa = 0$, on the other hand, the field $H_{c2} = \infty$, since the solution $\Psi_0 = 1$ exists in any field. For $\kappa \ll 1$, by using the method of successive approximations,[1] one can show, that on the boundary $z = 0$ the function $\Psi_0(0)$ is equal to

$$\Psi_0(0) = 1 - \varkappa h_0^2/2\sqrt{2} - 9\,(\varkappa h_0^2)^2/16 - \cdots. \qquad (3.14)$$

Hence, it is evident, that the expansion variable is the quantity κh_0^2, and Eq. (3.14) is valid so long as $1 - \Psi_0(0) \ll 1$. For this last reason, the absence of a maximum in the curve of Eq. (3.14) at $\Psi_0(0, h)$, corresponding to the field h_{c2}, means that the value $\Psi_0(0, h_{c2}) \equiv \Psi_{c2}$ is substantially different from unity.

On the other hand, for $\Psi_0(0) \to 0$, as can be seen from Eq. (3.9), $h_0 \to 1/\sqrt{2}$ and, thus, the curve of $\Psi_0(0, h_0)$ must have a maximum. From Eq. (3.14), and from an analysis of Eqs. (3.5) and (3.6) with the introduction of the variables $\zeta = \sqrt{\kappa}\xi$, $\chi = \Psi/\sqrt{\kappa}$ and $b = \sqrt{\kappa}\,a$, it can be inferred that, for $\sqrt{\kappa} \ll 1$ and $h_{c2} = \text{const}/\sqrt{\kappa}$, i.e.,

$$H_{c2}/H_{cb} = \sqrt{2}\,h_{c2} = 0.89/\sqrt{\varkappa}, \qquad (3.15)$$

where the constant was obtained by numerical integration of the equations, with $\kappa = 0.02$, for $H_{c2}/H_{cb} = 6.28$ and $\Psi_{c2} = 0.73$.

By way of example, $\Psi_0(0, h_0)$ has been plotted for $\kappa = 0.165$ in Fig. 5. The values of H_{c2}/H_{cb}, obtained by means of numerical calculation, are shown in Fig. 6. We also tabulate the values of H_{c2}/H_{cb} and Ψ_{2c} for some values of κ, as follows.

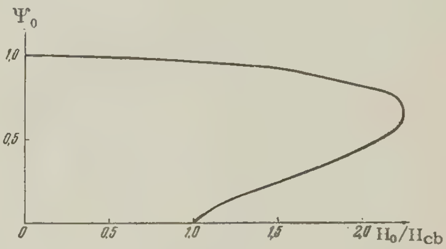

FIG. 5

$\varkappa =$	0.10	0.165	0.4	0.5	0.6	0.7	0.8
$H_{c2}/H_{cb} =$	2.87	2.25	1.64	1.52	1.45	1.39	1.34
$\Psi_{c2} =$	0.72	0.67	0.65	0.63	0.62	0.615	0.61

Experimentally it is difficult to observe superheating of the superconducting phase in bulk samples, but for samples of "average" dimensions (see the introduction) one can expect a different situation. Therefore, in particular, the thought arises that Hein and Steele observed not the equilibrium value of H_c for small superconducting cadmium spheres, but rather some value H_2 lying in the interval $H_c < H_2 < H_{c2}$. Indeed, by considering the transition to be an equilibrium one, in the value obtained in Ref. 8, $\delta_{00} = 8.8 \times 10^{-4}\,\text{cm}$, was two orders of magnitude larger than the quantity δ_{00} in Sn, Al, and other superconductors. Hence, as $T \to 0$, if $H_{cb} = 27.6$, we obtain $\kappa_0 = 2.16 \times$

$10^7 H_{cb}\delta_{00}^2 \approx 460$, which seem unlikely ($\kappa_0 = 0.17$ for Sn and $\kappa_0 = 0.05$ for Al). Recent measurements by Chaikin* on Cd confirm directly the incorrectness of the conclusion drawn by Hein and Steele, since Chaikin's data lead to $\delta_{00} = (10.2$ to $11.5) \times 10^{-6}$ and $\kappa_0 = 0.07$. In this connection it seems natural to assume that superheating was observed in Ref. 8, especially since only the destruction of superconductivity in a given field with increasing temperature was investigated. Even for the smallest spheres, like those investigated in Ref. 8, the condition $\kappa_0 a/\delta_{00} \gg 1$ is satisfied. Making use, therefore, of Eq. (3.15) or of the data presented in Fig. 6, we obtain† $H_{c2}/H_{cb} \approx 2.3$ at $T = 0$ whereas experimentally $H_2/H_{cb} \approx 2$. However, in Ref. 8, the field H_2/H_{cb} increased with increasing temperature by nearly a factor of four, whereas

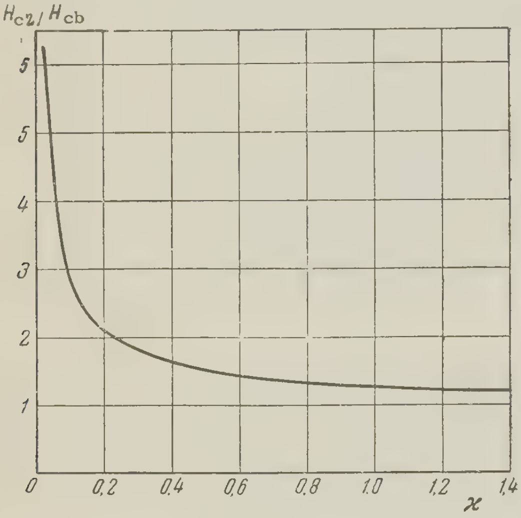

FIG. 6

$$\varkappa(T) = \varkappa_0 [1 + (T/T_c)^2]^{-1}, \qquad (3.16)$$

i.e., the field H_{c2}/H_{cb} increases only by approximately a factor of 1.4. On the other hand, for temperatures attained near T_c, the parameter $\kappa a/\delta_0$ equals approximately 3 or 4, i.e., it is not large enough to make use of the values H_{c2} obtained for the superconducting half-space.

Thus, the assumption of superheating still cannot be considered as contradicting the theory. In view of all this, further investigation of the problem of the boundary regions of superheating and supercooling, in particular in samples of "average" dimensions, is of interest.

Note added in proof (December 19, 1957). In November, 1957, we received in Moscow the manuscript of the detailed work of Bardeen, Cooper, and Schrieffer (henceforth BCS) in which a microscopic theory of superconductivity is formulated. It is shown that the London equation, generally speaking, does not hold even in a weak field, and the current is connected with the field in an integral way, as was already proposed by Pippard earlier. However, in the vicinity of T_c, subject to the condition $\xi_0/\delta_0(T) \lesssim 1$, the London equation, is still good to within an accuracy of not less than 10 or 15%, which is about the accuracy claimed by the BCS theory in its simplest version. The parameter is $\xi_0 = 0.18\hbar v_0/kT_c$, where v_0 is the velocity at the Fermi surface. For tin, $\xi_0 = 2.5 \times 10^{-5}$ cm, and the London equation can be used in a region of about 0.1 to 0.15° near T_c (this conclusion pertains directly only to bulk metals, but probably has a more general significance).

In the paper above, we have leaned on the work of Ref. 1 which transforms to the London theory in weak fields $H \ll H_c$. In view of what has been said, that work, and the results obtained on the basis thereof, can be quantitatively true only near T_c, as was assumed in Ref. 1, albeit for different reasons. In the general case, as one may believe and as is indicated at the end of the BCS article, the theory of Ref. 1 and its general conclusions probably retain a certain significance, but on the whole this question remains open.

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*The work was reported at the Fourth All-Union Conference on Low-Temperature Physics (July, 1957).

†Since the field on the surface of the sphere reaches 3/2 of its value at infinity, the quantity H_{c2} for the sphere is taken to be smaller by a factor of 2/3 than was calculated above for a superconductor with a plane interface.

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SOVIET PHYSICS JETP VOLUME 34(7), NUMBER 1 JULY, 1958

MOTION OF IONS IN A MIXTURE OF ISOTOPES

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Expressions are derived for the drift velocity of ions of isotopes in a mixture of isotopes. The principal interaction between the ions and atoms is assumed to be pure charge exchange.

In connection with the problem of the distribution of isotopes in a direct current discharge, the question of the mobility of an ion of an isotope in an isotopic mixture becomes of interest. In view of the possibility of charge exchange between an ion of one isotope and an atom of another isotope, Blank's rule for the mobility of ions in a mixture is not applicable in the given case.

Let there be a mixture of two isotopes with concentrations of neutral atoms N_1 and N_2. We shall denote the Maxwellian velocity distributions of the atoms by $n_1(v)$ and $n_2(v)$. Let the concentration of ions be N_1^+ and N_2^+ and their velocity distribution functions be $f_1(\mathbf{v})$ and $f_2(\mathbf{v})$. As is usual in problems on mobility, we shall disregard the effect of the ions on the velocity distribution function of the atoms and the interaction of the ions among themselves. The chief process of interaction between the ions and atoms is, in the given case, the exchange of charge without an exchange of momentum (pure charge exchange model), for which the charge-exchange cross section $q(u)$ can be considered the same in all four processes A^+A, A^+B, B^+B and B^+A. In the presence of a constant homogeneous electric field E, directed along the z axis, the velocity distribution function for the ions is found from a system of two kinetic equations, the first of which has the form

$$\frac{eE}{M_1} N_1^+ \frac{\partial f_1}{\partial v_z} = N_1 N_1^+ \int uq(u)\,[n_1(v)\,f_1(\mathbf{v}') - f_1(\mathbf{v})\,n_1(v')]\,d\mathbf{v}' + N_1 N_2^+ \int uq(u)\,n_1(v)\,f_2(\mathbf{v}')\,d\mathbf{v}' - N_1^+ N_2 \int uq(u)\,f_1(\mathbf{v})\,n_2(v')\,d\mathbf{v}', \qquad (1)$$

where M_1 is the mass of an atom of the first isotope and $u = |\mathbf{v} - \mathbf{v}'|$. The second and third terms on the right describe the appearance of A ions as a result of impacts of the type B^+A and their disappearance upon impacts of the type A^+B. The second kinetic equation is obtained from Eq. (1) by an interchange of indices.

Let us solve the system of kinetic equations in the limiting cases of weak and strong fields. For small fields, when the energy acquired by an ion over a mean free path is much less than the thermal energy, we apply the method of Langevin, who assumes the velocity distribution of the ions to be Maxwellian with a small superimposed drift in the direction of the field:

$$f_1(\mathbf{v}) = A_1 \exp\left\{-\frac{M_1}{2kT}\,[v_x^2 + v_y^2 + (v_z - v_1)^2]\right\} \approx n_1(v) + \frac{M_1}{kT}\,v_1 v_z n_1(v). \qquad (2)$$

An analagous expression is obtained for $f_2(\mathbf{v})$.

The law of conservation of the number of particles is obtained from Eq. (1) by integration over the velocities. As a result, using Eq. (2), we obtain

$$N_1^+/N_1 = N_2^+/N_2. \tag{3}$$

This result is a consequence of the fact that, in our setting up of the problem the formation and disappearance of ions proceeds, only with charge exchange and that the charge-exchange cross sections for impacts of the type A^+B and B^+A are assumed equal. The unknown drift velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are determined from the momentum balance equations that are obtained if each of the kinetic equations is multiplied by $v_z d\mathbf{v}$ and integrated over the velocities. After simple calculations, using Eqs. (2) and (3), we obtain a system of two equations, the first of which has the form

$$\frac{3\sqrt{\pi}}{16}\frac{eE}{M_1}\sqrt{\frac{M_1+M_2}{2kT}}$$

$$= v_1\left[N_1P_1\sqrt{\frac{1+\gamma_{21}}{2}} + \frac{3}{4}N_2R\sqrt{\gamma_{12}} + \frac{1}{2}N_2P\sqrt{\gamma_{21}}\right]$$

$$+ v_2N_2\sqrt{\gamma_{21}}\left[\frac{1}{2}P - \frac{3}{4}R\right]. \tag{4}$$

The second equation is obtained from the first by an interchange of indices. Here the following notation has been introduced:

$$\gamma_{12} = \frac{1}{\gamma_{21}} = \frac{M_1}{M_2}; \quad P = \frac{1}{2}\int_0^\infty e^{-x}x^2q(u)\,dx;$$

$$R = \int_0^\infty e^{-x}\,xq(u)\,dx; \qquad x = \frac{M_1M_2}{M_1+M_2}\,\frac{u^2}{2kT}.$$

P_1 and P_2 are obtained from P by replacing the reduced mass by $M_1/2$ and $M_2/2$ respectively. The expressions for v_1 and v_2 are easy to obtain from the system of Eqs. (4), but we shall not write them out because of their unwieldiness. For the case where the charge exchange cross section can be regarded as independent of velocity, $P_1 = P_2 = P = R = q$, and we obtain

$$v_1 = \frac{3\sqrt{\pi}}{16}\,\frac{eE}{M_1q(N_1+N_2)\sqrt{kT/M_1}}\,\frac{(4\sqrt{(1+\gamma_{21})/2}+1)\,c_2 + (2+3\gamma_{21})\,c_1}{c_1^2(2+3\gamma_{21}) + c_2^2(3\sqrt{\gamma_{12}}+2\sqrt{\gamma_{21}}) + 2^{-1/2}c_1c_2\;3\sqrt{\gamma_{12}}(1+\gamma_{21})+4](1+\gamma_{21})^{1/2}}, \tag{5}$$

where $c_1 = N_1/(N_1 + N_2)$, $c_2 = N_2/(N_1 + N_2)$.

We obtain the expression for v_2 from Eq. (5) by an interchange of indices. In the special cases where $c_2 = 0$ or $M_1 = M_2$, Eq. (5) goes over to the ordinary formula for the drift velocity of ions in their respective gas.[1] For the case where $c_1 \to 0$,

$$v_1 = \frac{3\sqrt{\pi}}{16}\,\frac{eE}{M_1qN_2\sqrt{kT/M_2}}\,\frac{4\sqrt{(1+\gamma_{21})/2}+1}{3+2\gamma_{21}}. \tag{6}$$

In the general case for the ratio of the drift velocities of ions of two isotopes, we obtain

$$\frac{v_1}{v_2} = \frac{(4\sqrt{(1+\gamma_{21})/2}+1)\,c_2 + (2+3\gamma_{21})\,c_1}{(4\sqrt{(1+\gamma_{12})/2}+1)\,c_1 + (2+3\gamma_{12})\,c_2}. \tag{7}$$

From Eq. (7) it is possible to find the limits within which the ratio of the drift velocities of ions in an isotopic mixture varies with a change in concentration from $c_2 = 0$ to $c_1 = 0$. For example, for a mixture of hydrogen with deuterium we find that v_1/v_2 lies between the limits 1.8 and 1.7; for a mixture of He^3 with He^4, it is within the limits 1.27 and 1.25. The results obtained refute the assertion to the effect that as a result of frequent charge exchanges a homogeneous drift velocity is set up for ions of both isotopes.[2] The drift velocity of an ion of the lighter isotope noticeably exceeds the drift velocity of an ion of the heavier isotope.

One frequently encounters the case where the mixture contains a third component besides atoms of isotopes (for example, molecules of isotopes). Here, the interaction of ions with particles of the third component does not have charge-exchange character (the interaction depends essentially upon polarization forces). In this case the kinetic equation (1) contains additional terms on the right side, which take into account collisions of ions of isotopes with particles of the third component. Using Langevin's method, we obtain a system of two linear equations in v_1 and v_2, the first of which differs from Eq. (4) by the additional term

$$\frac{3\sqrt{\pi}}{16}\sqrt{\frac{M_1+M_2}{2kT}}\,\frac{e}{M_1}\,\frac{v_1}{b_{13}}. \tag{8}$$

Here b_{13} is the mobility of an ion of the first isotope in the third component (in the Langevin approximation):

$$b_{13} = \frac{e}{kT}\,\frac{3\sqrt{\pi}}{16N_3}\sqrt{\frac{2kT}{M_1M_3}}\,(M_1+M_3)\,\frac{1}{L};$$

$$L = \frac{1}{2}\int_0^\infty Q_{13}(u)\,x^2e^{-x}dx; \qquad x = \frac{M_1M_3}{M_1+M_3}\,\frac{u^2}{2kT};$$

N_3 and M_3 are the concentration and mass of particles of the third component and Q_{13} is the diffusion interaction cross section for ions of the first isotope with particles of the third component. The second equation for v_1 and v_2 is obtained from the first, as usual, by the interchange of the indices 1 and 2. In the presence of several additional components, each of them gives an additional term of the type of Eq. (8) in the right side of Eq. (4).

For the case of strong fields where it is possible to disregard the thermal motion of the atoms we obtain, by a method completely analagous to that applied in Ref. 3, an exact result for the drift velocity of an ion of an isotope in an isotopic mixture

$$v_1 = [2eE / \pi M_1 (N_1 + N_2) q]^{1/2}. \qquad (9)$$

This result is obtained under the assumption that the charge exchange cross section is independent of velocity (in the method applied this need not necessarily be so, and is assumed for simplicity). The drift velocities in this case are inversely proportional to the square roots of the masses of the ions. This conclusion is physically obvious inasmuch as in the assumption of quiescent atoms the difference in their masses plays no role.

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POLARIZATION OF SLOW NEUTRONS SCATTERED IN CRYSTALS

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Formulas are derived for the polarization and cross section of slow neutrons scattered in crystals with polarized nuclei.

ROSE[1] has examined the polarization effects arising in the scattering of slow neutrons in crystals, and has obtained formulas for the scattering cross section of polarized neutrons in crystals with polarized nuclei. He considered only crystals of atoms with a single isotope. In the present note, his results are generalized to include crystals of several isotopes, and expressions obtained for the change in neutron polarization due to scattering.

It is well known[2] that the scattering amplitude for slow neutrons in a crystal is proportional to the quantity

$$\left(f\mu' \left| \sum_j e^{i\mathbf{q}\mathbf{R}_j} \left(A_j + \frac{1}{2} B_j \sigma \mathbf{I}_j\right) \right| i\mu\right) = F, \qquad (1)$$

where i and f are the initial and final states of the scatterer; μ and μ' are the initial and final neutron spin projections; $\sigma/2$ is the neutron spin; $\mathbf{q}$ is the momentum transferred to the scatterer; $\mathbf{R}_j$ and $\mathbf{I}_j$ are the coordinates and spins of the j-th nucleus; and A_j, B_j are complex constants. The summation is over all the nuclei in the crystal.

From (1) it is easy to obtain the following expression for the coherent and incoherent parts of the scattering cross section, averaged over all orientations of the neutron and nuclear spins, and over all possible distributions of isotopes in the lattice:

$$\sigma_{\mathrm{coh}} = \sigma_{\mathrm{coh}}^{(0)} \{1 + |\langle A\rangle|^{-2} [(\mathbf{pN}) \operatorname{Re} \langle A^*\rangle \langle BIP\rangle + {}^1/_4 |\langle BIP\rangle|^2]\}; \qquad (2)$$

$$\sigma_{\text{incoh}} = \sigma^{(0)}_{\text{incoh}}\{1 + [\langle |A|^2\rangle - |\langle A\rangle|^2$$

$$+ {}^1/_4\langle |B|^2 I(I+1)\rangle]^{-1}[(\mathbf{pN})\,\mathrm{Re}\,(\langle A^*BIP\rangle - \langle A^*\rangle\langle BIP\rangle)$$

$$- {}^1/_4(\mathbf{pN})\langle |B|^2 IP\rangle - {}^1/_4|\langle BIP\rangle|^2]\}, \tag{3}$$

Here $\mathbf{p}$ is the polarization vector of the neutron, $\mathbf{N}$ is a unit vector defining the direction in which the nuclei are polarized, P is their polarization, $\langle\ldots\rangle$ means an average over all distributions of isotopes in the lattice, and $\sigma^{(0)}$ is the coherent or incoherent cross section for unpolarized nuclei. It is clear that formulas (2) and (3) hold for both the differential and total scattering cross sections. If the scatterer consists of one isotope only, (2) and (3) reduce to the expressions obtained by Rose.

It is not difficult to compute the polarization of the neutrons after they have been scattered in a definite direction. We use the formula

$$\mathbf{p}_1 = \mathrm{Sp}\,(F^+\sigma F\rho)\,/\,\mathrm{Sp}\,(F^+F\rho),$$

where ρ is the density matrix describing the polarization of the neutrons before scattering. We obtain the formula

$$\mathbf{p}_1 = (\sigma_{\text{coh}} + \sigma_{\text{incoh}})^{-1}\{\sigma^{(0)}_{\text{coh}}[\mathbf{p} + |\langle A\rangle|^{-2}(\mathbf{N}\,\mathrm{Re}\,\langle A^*\rangle\langle BIP\rangle$$

$$+ \mathrm{Im}\,\langle A^*\rangle\langle BIP\rangle[\mathbf{Np}] + {}^1/_4|\langle BIP\rangle|^2[2(\mathbf{Np})\mathbf{N} - \mathbf{p}])]$$

$$+ \sigma^{(0)}_{\text{incoh}}[\mathbf{p} + (\langle |A|^2\rangle - |\langle A\rangle|^2 + {}^1/_4\langle |B|^2 I(I+1)\rangle)^{-1}$$

$$\times\{\mathbf{N}\,\mathrm{Re}\,(\langle A^*BIP\rangle - \langle A^*\rangle\langle BIP\rangle) + [\mathbf{Np}]\,\mathrm{Im}\,(1^*BIP\rangle$$

$$- \langle A^*\rangle\langle BIP\rangle) + {}^1/_2\langle |B|^2\hat{D}\rangle\mathbf{p} + {}^1/_4\langle |B|^2 IP\rangle\mathbf{N}$$

$$- {}^1/_4|\langle BIP\rangle|^2[2(\mathbf{pN})\mathbf{N} - \mathbf{p}]\}]\}. \tag{4}$$

In this formula, $\hat{D}$ is a tensor related to the polarization tensor $\hat{T}$ of the nuclei by the equation

$$D_{\alpha\beta} = \frac{1}{3}I(2I-1)\left(T_{\alpha\beta} - 2\frac{I+1}{2I-1}\,\delta_{\alpha\beta}\right), \tag{5}$$

where

$$T_{\alpha\beta} = \frac{3}{2I(2I-1)}\left(\overline{I_{j\alpha}I_{j\beta} + I_{j\beta}I_{j\alpha}} - \frac{2}{3}I(I+1)\,\delta_{\alpha\beta}\right), \tag{6}$$

and $\hat{D}\mathbf{p}$ is a vector with components $(\hat{D}\mathbf{p})_{\alpha\beta} = D_{\alpha\beta}p_\beta$. The bar means average over spin projections.

If the incident neutron had not been polarized, then its polarization after scattering would be

$$\mathbf{p}_1 = \mathbf{N}\,(\sigma_{\text{coh}} + \sigma_{\text{incoh}})^{-1}\{\sigma^{(0)}_{\text{coh}}|\langle A\rangle|^{-2}\,\mathrm{Re}\,\langle A^*\rangle\langle BIP\rangle$$

$$+ \sigma^{(0)}_{\text{incoh}}\frac{\mathrm{Re}\,(\langle A^*BIP\rangle - \langle A^*\rangle\langle BIP\rangle) + {}^1/_4\langle |B|^2 IP\rangle}{\langle |A|^2\rangle - |\langle A\rangle|^2 + {}^1/_4\langle |B|^2 I(I+1)\rangle}\}. \tag{7}$$

If $\sigma_{\text{coh.}} \gg \sigma_{\text{incoh.}}$, which will hold, for example, in the direction of a Bragg peak,

$$\mathbf{p}_1 = \mathbf{N}\,\frac{\mathrm{Re}\,\langle A^*\rangle\langle BIP\rangle}{|\langle A\rangle|^2 + {}^1/_4|\langle BIP\rangle|^2}. \tag{8}$$

If, on the other hand, $\sigma_{\text{coh.}} \ll \sigma_{\text{incoh.}}$, which will hold, for example for very slow neutrons,[2] then

$$\mathbf{p}_1 = \mathbf{N}\,\frac{\mathrm{Re}\,(\langle A^*BIP\rangle - \langle A^*\rangle\langle BIP\rangle) + {}^1/_4\langle |B|^2 IP\rangle}{\langle |A|^2\rangle - |\langle A\rangle|^2 + {}^1/_4\langle |B|^2 I(I+1) - {}^1/_4\langle BIP\rangle|^2}. \tag{9}$$

Expression (9) reduces to

$$\mathbf{p}_1 = \mathbf{N}\,P/[I(1 - P^2) + 1]. \tag{10}$$

if all the scatterer atoms are of one isotope.

If the nuclei are unpolarized, but are oriented ($P = 0$, $\hat{T} \neq 0$), then

$$\mathbf{p}_1 = \frac{\sigma^{(0)}_{\text{incoh}}\langle |B|^2\hat{D}\rangle\mathbf{p}}{2(\sigma^{(0)}_{\text{coh}} + \sigma^{(0)}_{\text{incoh}})[\langle |A|^2\rangle - |\langle A\rangle|^2 + {}^1/_4\langle |B|^2 I(I+1)\rangle]}. \tag{11}$$

If, in addition, $\sigma_{\text{incoh.}} \gg \sigma_{\text{coh.}}$ and the scatterer consists of one isotope only, then

$$\mathbf{p}_1 = 2\hat{D}\mathbf{p}/I(I+1). \tag{12}$$

In conclusion, the author would like to express his gratitude to I. M. Shmushkevich, who suggested this problem.

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THE RADIATION FROM AN ELECTRON MOVING IN A MAGNETOACTIVE PLASMA

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We consider the spectral and angular distribution of the energy radiated by an electron moving in a magnetoactive plasma, and find the polarization of the radiation.

1. The problem of radiation of electromagnetic energy by charged particles moving in a magnetic field is of considerable interest for some parts of astrophysics and radioastronomy. But in most papers on this subject, the treatment is limited to the case of motion of electrons in vacuum.[1-3]

The presence of the medium can, under certain conditions, strongly influence the character of the radiation and, for example, lead to a reduction in intensity of the low frequency radiation from discrete sources.[2,4]

The problem of the radiation from an electron moving in a magnetic field in an isotropic medium was treated by Tsytovich[5] and Razin.[4] However, the magnetic field which accelerates the electron also causes the medium in which the electron moves to become magnetoactive. The anisotropy which arises in this way can be large in the case of a plasma. Under these conditions we must solve the problem of the radiation from an electron moving in a gyrotropic medium. This problem is solved in the present paper using the Hamiltonian method.

We should mention that this method was used by Ginzburg,[6] and later by Kolomenskii,[7] to determine the energy of the radiation from charged particles in anisotropic media.

2. The field equations for a charge e, moving along a vortex line in a magnetoactive medium, have the following form:

$$\operatorname{curl}\mathbf{H}=\frac{4\pi}{c}e\mathbf{v}\delta(\mathbf{r}-\mathbf{r}_e)+\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t},\qquad \operatorname{div}\mathbf{H}=0,$$
$$\operatorname{div}\mathbf{D}=4\pi e\delta(\mathbf{r}-\mathbf{r}_e),\quad \operatorname{curl}\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{H}}{\partial t},\tag{1}$$

where the components of the radius vector $\mathbf{r}_e$ and the velocity $\mathbf{v}$ of the electron along the x, y, z axes are, respectively,

$$r_0\cos\Omega_0 t;\qquad r_0\sin\Omega_0 t;\qquad v_2 t$$
$$\text{and}\ -v_1\sin\Omega_0 t;\quad v_2\cos\Omega_0 t;\quad v_2$$

($v_1=\Omega_0 r_0$). Here it is assumed that the axis of the vortex line, which is parallel to the constant magnetic field $\mathbf{H}_0$, is along the z axis, while $\mathbf{v}_1$, $\mathbf{v}_2$ are the projections of the electron velocity on the xy plane and the z axis. The Fourier components of the electric induction $\mathbf{D}_\omega$ and the electric field intensity $\mathbf{H}_\omega$ are related by

$$\mathbf{D}_\omega=\{\varepsilon_{\alpha\beta}(\omega)\}\mathbf{E}_\omega,$$

where $\{\epsilon_{\alpha\beta}\}$ is the Hermitian dielectric tensor ($\epsilon_{\alpha\beta}=\epsilon^*_{\beta\alpha}$), whose components in the system of coordinates which we are using are given on p. 326 of Ref. 8. According to the Hamiltonian method,[6,7] the vector potential of the radiation field,

$$\mathbf{A}=\sqrt{4\pi}\,c\sum_{j\lambda}q_{j\lambda}\mathbf{a}_{j\lambda}e^{i\mathbf{k}_\lambda\mathbf{r}}\tag{2}$$

can be found by solving the system of oscillator equations

$$q_{j\lambda}+\omega^2_{j\lambda}q_{j\lambda}=\sqrt{4\pi}\frac{e\mathbf{v}}{n_{j\lambda}}\mathbf{a}^*_{j\lambda}e^{-i\mathbf{k}_\lambda\mathbf{r}_e}.\tag{3}$$

In these relations, $\mathbf{a}_{j\lambda}$ is the complex polarization vector (the index $j = 1, 2$ corresponds to the two normal waves propagating in the gyrotropic plasma). The wave vector $\mathbf{k}_\lambda$ is related to the frequency $\omega_{j\lambda}$ by $\omega^2_{j\lambda}=k^2_\lambda c^2/n^2_{j\lambda}$, where $n_{j\lambda}$ is the refractive index. If we introduce the parameters α_j and β_j through the relations

$$\alpha_j=K_j\cos\theta+\gamma_j\sin\theta,\quad \beta_j=\gamma_j\cos\theta-K_j\sin\theta,$$
$$iK_j=E_\theta/E_x;\qquad i\gamma_j=E_{rj}/E_x,\tag{4}$$

the polarization vector $\mathbf{a}_{j\lambda}$ of the j-th normal mode will have the following components along the coordinate axes: $1/\sqrt{2}$, $i\alpha_j/\sqrt{2}$; $i\beta_j/\sqrt{2}$. The orientation of E_x, H_θ, E_r is clear from Fig. 1.

For a magnetoactive plasma, the quantities K_j and n_j are given by the expressions

$$n_j^2 = 1 - 2V(1-V)/[2(1-V)$$

$$- u \sin^2\theta \pm \sqrt{u^2 \sin^4\theta + 4u(1-V)^2 \cos^2\theta}];$$

$$K_j = \frac{-2\sqrt{u}(1-V)\cos\theta}{[u\sin^2\theta \pm \sqrt{u^2\sin^4\theta + 4u(1-V)^2\cos^2\theta}]} \cdot \quad (5)$$

On the basis of the definition (4) we can also easily compute the parameter γ_j:

$$\gamma_j = \frac{-\sin\theta V\sqrt{u} + K_j uV\cos\theta\sin\theta}{1 - u - V(1 - u\cos^2\theta)}. \quad (6)$$

In formulas (5) and (6) we assume, as usual, that

$$V = 4\pi e N / m\omega^2; \qquad \sqrt{u} = eH_0 / mc\omega,$$

where m is the electron mass and N is the electron concentration in the plasma. Using the expression for the components of the vector $\mathbf{a}_{j\lambda}$, and the expansion

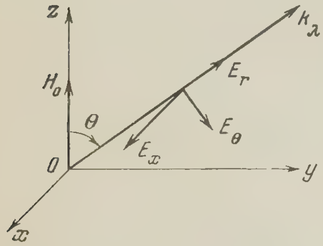

FIG. 1. $E_x \perp H_0$; E_θ, E_r and k_λ are in the yz plane; $E_x \perp k_\lambda$, $E_\theta \perp k_\lambda$.

$$\exp\{-ik_\lambda r_0 \sin\theta \sin\Omega_0 t\} = \sum_{-\infty}^{\infty} J_s(k_\lambda r_0 \sin\theta)\, e^{-is\Omega_0 t}$$

(J_s is the Bessel function of order s), we reduce Eq. (3) to the following form:

$$\ddot{q}_{j\lambda} + \omega_{j\lambda}^2 q_{j\lambda} =$$

$$-\sqrt{4\pi}\frac{ei}{n_{j\lambda}} \sum_{-\infty}^{\infty} G_s(k_\lambda r_0 \sin\theta) \exp\{-it[s\Omega_0 + v_2 k_\lambda \cos\theta]\}. \quad (7)$$

Here

$$G_s(k_\lambda r_0 \sin\theta) = v_1 J_s'(k_\lambda r_0 \sin\theta)$$

$$+ \left[\frac{\alpha s v_1}{k_\lambda r_0 \sin\theta} + \beta v_2\right] J_s(k_\lambda r_0 \sin\theta),$$

while J_s' is the derivative of the Bessel function with respect to its argument.

In the system of equations (7), those frequencies $\omega_{j\lambda}$ which satisfy the relation

$$\omega_{j\lambda} \approx s\Omega_0 + v_2 k_\lambda \cos\theta = p_\lambda, \quad (8)$$

give solutions of (7) which increase with time and correspond to radiation. The equations for frequencies given by (8) are:

$$\ddot{q}_{j\lambda} + \omega_{j\lambda}^2 q_{j\lambda} = -\sqrt{4\pi}\frac{ei}{n_{j\lambda}} G_s e^{-ip_\lambda t} = f(t). \quad (9)$$

The solution of this system for the initial conditions $t = 0$; $q_{j\lambda} = \dot{q}_{j\lambda} = 0$ is elementary:

$$q_{j\lambda} = \frac{i}{2\omega_{j\lambda}}\left[e^{-i\omega_{j\lambda}t}\int_0^t f(t)\, e^{i\omega_{j\lambda}t}\, dt - e^{i\omega_{j\lambda}t}\int_0^t f(t)\, e^{-i\omega_{j\lambda}t}\, dt\right].$$

The energy of the oscillator $q_{j\lambda}$ after time T is

$$W_{j\lambda} = \frac{1}{4}\left|\int_0^T f(t)\, e^{i\omega_{j\lambda}t}\, dt\right|^2 + \frac{1}{4}\left|\int_0^T f(t)\, e^{-i\omega_{j\lambda}t}\, dt\right|^2.$$

Summing over all oscillators $q_{j\lambda}$ in the frequency interval $d\omega_{j\lambda}$ and solid angle $d\Omega$, we find that the energy radiated within this range of frequency and angle for the s-th harmonic is given by the following expression:

$$W_{sj} = \frac{1}{2}|G_s|^2 \frac{1 - \cos T[-s\Omega_0 + \omega_{j\lambda}(1 - \beta_2 n_{j\lambda}\cos\theta)]}{[-s\Omega_0 + \omega_{j\lambda}(1 - \beta_2 n_{j\lambda}\cos\theta)]^2}\rho_{j\lambda} d\omega_{j\lambda} d\Omega, \quad (10)$$

where

$$\beta_2 = v_2/c; \qquad \rho_{j\lambda} = n_{j\lambda}^3 \omega_{j\lambda}^2/(2\pi c)^3.$$

For T → ∞, (10) becomes a δ function of the argument

$$-s\Omega_0 + \omega_{j\lambda}(1 - \beta_2 n_{j\lambda}\cos\theta).$$

This means that only the Doppler frequency ω_s, defined by the equation

$$-s\Omega_0 + \omega_s(1 - \beta_2 n_{js}\cos\theta) = 0, \quad (11)$$

is responsible for the harmonic $\Omega_0 s$. Using (10) it is easy to find the total energy of the radiation for the j-th normal mode, as a sum:

$$W_j = \sum_{-\infty}^{\infty} W_{sj}.$$

3. We should mention that, according to formula (10), radiation is also possible for $s = 0$ (zeroth harmonic). In this case the energy is radiated at the Cerenkov angle θ, which satisfies the equation:

$$1 - \beta_2 n_j(\omega;\ \theta_1)\cos\theta_1 = 0,$$

where, for simplicity, we have dropped the subscripts on the frequency $\omega_{j\lambda}$. Carrying out the integration over solid angle in (10), we find that the radiated energy in the j-th normal mode is ($s = 0$):

$$W_{0j} = \frac{Te^2}{2c^2 v_2} \int\limits_{\beta_2 n_j(\omega) \geqslant 1} \frac{\omega d\omega \left\{ v_1 J_0' \left(\frac{v_1}{v_2} \frac{\omega}{\Omega_0} \sqrt{\beta_2^2 n_j^2 - 1} \right) + \beta_j v_2 J_0 \left(\frac{v_1}{v_2} \frac{\omega}{\Omega_0} \sqrt{\beta_2^2 n_j^2 - 1} \right) \right\}^2}{\left| 1 - (\partial n_j / \partial \theta) / n_j \sqrt{\beta_2^2 n_j^2 - 1} \right|}. \tag{12}$$

If the electron moves uniformly along the z axis ($v_1 = 0$; $r_0 = 0$), the preceding expression reduces to formula (4.5) of Ref. 7 for Cerenkov radiation:

$$W_{0j} = \frac{Te^2 v_2}{2c^2} \int\limits_{\beta_2 n_j \geqslant 1} \frac{\beta^2 \omega d\omega}{\left| 1 - (\partial n_j / \partial \theta) / n_j \sqrt{\beta_2^2 n_j^2 - 1} \right|}. \tag{13}$$

If is necessary to remark that formulas (12) and (13) are invalid for those frequencies for which the refractive index $n_j(\omega) \to \infty$, since in this case the expression for W_{0j} diverges ($\beta_j \to \infty$ like n_j^2). In order to eliminate this divergence, we must either take account of collisions in the plasma (which will make $n_j(\omega)$ finite everywhere), or set some limiting value n_m for the refractive index n_j, if we are interested in the total loss from the moving particle. In this case, the limits of the domain of integration will be given by the following inequalities (for more details, cf. Ref. 9):

$$\beta_2 n_m \geqslant \beta_2 n_j(\omega) \geqslant 1.$$

This remark applies to all relations obtained later on, when $n_j(\omega, \theta)$ goes to infinity.

4. For harmonics other than the zeroth, it is more convenient to carry out the integration over ω in (10). Then the energy radiated, at the frequency ω_s defined by (11), in time T into the solid angle $d\Omega$ is

$$W_{js} = \frac{Te^2 \omega_s^2 n_j d\Omega \left\{ v_1 J_s' (n_j \omega_s r_0 \sin\theta / c) + [\alpha s v_1 / k(\omega_s) r_0 \sin\theta + \beta v_2] J_s(n_j \omega_s r_0 \sin\theta / c) \right\}^2}{4\pi c^3 \left| 1 - \beta_2 \cos\theta \left(n_j + \omega_s \partial n_j(\omega_s) / \partial\omega \right) \right|}. \tag{14}$$

If we set $u = V = 0$ ($n_j = 1$; $\alpha_j = \pm\cos\theta$) in (11), (14) and (5), i.e., if we treat the case of motion of the electron along a vortex line in vacuum, the amount of energy radiated will be given by the following:

$$W_s = \sum_{j=1}^{2} W_{js} = \frac{Te^2 (s\Omega_0)^2 d\Omega}{2\pi c^3 (1 - \beta_2 \cos\theta)^3} \left\{ v_1^2 \left[J_s' \left(\frac{s\beta_1 \sin\theta}{1 - \beta_2 \cos\theta} \right) \right]^2 + \left[\frac{c\cos\theta - v_2}{\sin\theta} \right]^2 J_s^2 \left(\frac{s\beta_1 \sin\theta}{1 - \beta_2 \cos\theta} \right) \right\},$$

where $\beta_1 = v_1/c$.

If, in (14), we set $v_2 = 0$, which corresponds to motion of the electron in a circle, we get the analog of the Schott formula for a gyrotropic medium:

$$W_{js} = \frac{Te^2 \Omega_0^2 s^2 n_j v_1^2}{4\pi c^3} \left[J_s'(s n_j \beta_1 \sin\theta) + \frac{\alpha_j}{\sin\theta \beta_1 n_j} J_s(s n_j \beta_1 \sin\theta) \right]^2 d\Omega. \tag{15}$$

For the energy radiated in vacuum ($n_j = 1$, $\alpha_j = \pm\cos\theta$), expression (15) gives the Schott formula (cf., for example, Ref. 10, p. 216):

$$W_s = \sum_{j=1}^{2} W_{js} = \frac{Te^2 \Omega_0^2 s^2}{2\pi c^3} \left\{ \cot^2\theta J_s^2(s\beta_1 \sin\theta) + \beta_1^2 J_s'^2 \right\} d\Omega.$$

5. Let us now examine the character of the radiation corresponding to high harmonics

$$s = (\omega_s / \Omega_0)(1 - \beta_2 n_j \cos\theta) \gg 1$$

(from now on we omit the subscript s on the frequency ω_s).

Using (11) and (14), we easily get the spectral density of the radiation, which is given by

$$W_{j\omega} d\omega = \frac{\pi T}{2\Omega_0} \left| G_s \left(\frac{\omega}{\Omega_0} \beta_1 n_j \sin\theta \right) \right|^2 \rho_{j\omega} d\omega d\Omega.$$

If there are angles θ, within the particular frequency interval, for which $\beta_j n_j \sin\theta$ is of order one, the Bessel functions in the preceding expression can be transformed to Airy functions (cf., for example, Ref. 10, p. 217). We then find that

$$W_{j\omega} d\omega = \frac{Te^2 \omega^2 n_j d\omega d\Omega}{(2c)^3 \pi^2 \Omega_0} \left(\frac{2\Omega_0}{\omega \beta_1 n_j \sin\theta} \right)^{4/3} \left\{ -\Phi'(x) + \left[\frac{v_1 \alpha (1 - \beta_2 n_j \cos\theta)}{\beta_1 n_j \sin\theta} + \beta v_2 \right] \left(\frac{\omega \beta_1 n_j \sin\theta}{2\Omega_0} \right)^{1/3} \Phi(x) \right\}^2, \tag{16}$$

where the Airy function is

$$\Phi(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos\left(x\xi + \frac{\xi^3}{3}\right) d\xi; \qquad x = (2\omega^2 / \beta_1 n_j \Omega_0^2 \sin\theta)^{1/3} (1 - \beta_1 n_j \sin\theta - \beta_2 n_j \cos\theta).$$

For motion of the electron in a circle $(v_2 = 0)$, we obtain from (16):

$$W_{j\omega} d\omega = \frac{Te^2 v_1^2 \omega^2 n_j d\Omega d\omega}{(2c)^3 \pi^2 \Omega_0} \left(\frac{2\Omega_0}{\omega\beta_1 n_j \sin\theta}\right)^{4/3} \left\{ -\Phi'(x) + \frac{\alpha_j}{\beta_1 n_j \sin\theta} \left(\frac{\omega\beta_1 n_j \sin\theta}{2\Omega_0}\right)^{1/3} \Phi(x) \right\}^2; \quad x = (2\omega^2 / \beta_1 n \Omega_0^2 \sin\theta)^{1/3} (1 - \beta_1 n \sin\theta). \tag{17}$$

Let us consider our result in a little more detail. To do this we give the asymptotic expression for the Airy function, whose behavior for positive arguments is essentially different from its behavior for negative arguments. Thus, if $x \gg 1$,

$$\Phi(x) = x^{-1/4} \exp\{-2x^{3/2}/3\},$$

whereas for $x < 0$; $|x| \gg 1$:

$$\Phi(x) = |x|^{-1/4} \sin\left(\frac{2}{3}|x|^{3/2} + \frac{\pi}{4}\right).$$

From this it follows that the character of the angular distribution of the energy radiated at frequency ω depends mainly on the nature of the roots of the equation

$$1 - \beta_1 n(\omega, \theta) \sin\theta = 0. \tag{18}$$

For example, for an isotropic medium $(n(\theta) = \text{const} > 1)$, Eq. (18) has two roots:

$$\sin\theta_1 = 1/\beta_1 n, \qquad \sin\theta_2 = \sin(\pi - \theta_1) = 1/\beta_2 n.$$

In this case the main part of the radiation is at angles θ satisfying the inequalities

$$\theta_1 \leqslant \theta \leqslant \theta_2, \tag{19}$$

and for these values of θ, the directivity pattern is multi-lobed, with the main lobes directed at angles close to θ_1 and θ_2. For values of θ outside the interval defined by (19), the radiation intensity drops exponentially as we move away from the angles θ_1 and θ_2.

For a gyrotropic plasma, $n(\theta)$ is given by the quite complicated expression (5), so that it is convenient to solve (18) graphically by finding the points of intersection of the functions $n(\theta)$ and $1/\beta_1 \sin\theta$. The following two cases are possible for the angular distribution of the energy:

(1) if

$$dn(\theta_1)/d\theta > -\cos\theta_1 / \beta_1 \sin^2\theta_1,$$

then for

$$\theta_1 \leqslant \theta \leqslant \pi - \theta_1 \tag{20}$$

$x < 0$, while for

$$\theta_1 > \theta, \qquad \theta > \pi - \theta_1, \tag{21}$$

$x > 0$, i.e. the radiation is mainly contained within the cone defined by the inequalities (19);

(2) if

$$dn(\theta_1)/d\theta < -\cos\theta_1 / \beta_1 \sin^2\theta_1,$$

then the radiation is concentrated at the angles corresponding to the inequalities (21).

Figure 2 shows the graph of the function

$$W_{1\omega}(\theta) \approx n_1 \left(\frac{2\Omega_0}{\omega\beta_1 n_1 \sin\theta}\right)^{4/3} \left\{ -\Phi'(x) + \frac{\alpha_j}{\beta_1 n_j \sin\theta} \left(\frac{\omega\beta_1 n_1 \sin\theta}{2\Omega_0}\right)^{1/3} \Phi(x) \right\}^2 = W(\theta),$$

defined by (17), for the following values of the parameters:

$$u = 0.25; \quad V = 0.8; \quad \beta_1 = 0.4; \quad \omega/\Omega_0 = 20.$$

As we see from the graph, the second case occurs here.

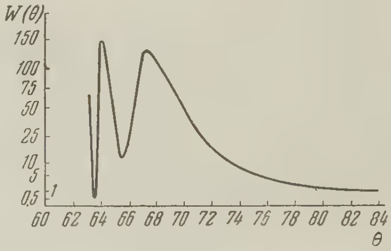

FIG. 2

We note that expression (17) for the energy W_j diverges for $n_j \to \infty$, since α_j tends to infinity like n_j^2 (in our example this occurs at $\theta = 60°$); whereas in an isotropic medium, the amount of energy radiated, W_ω, remains finite even for $n \to \infty$ [cf. the remark concerning divergences which we made with respect to forumlas (12) and (13)].

We now give the approximate value of (17) for the case where the given frequency interval contains angles θ for which

$$1 > \beta_1 n_j \sin\theta \approx 1.$$

It is easy to see that then $x > 0$ and $\Phi(x)$ has a sharp maximum for angles satisfying the relation

$$\beta_1 n_j \sin\theta \approx 1. \tag{22}$$

If we make use of the approximate equality (22), we get the following expression for the energy radiated during one revolution of the electron:

$$W_{\omega j}d\omega \approx (2e^2v_1^2/\pi c^3)(\omega/2\Omega_0)^{2/3} n_j d\omega d\Omega \{-\Phi'(x) + \alpha_j(\omega/2\Omega_0)^{1/3}\Phi(x)\}^2,$$

$$x = (2\omega^2/\Omega_0^2)^{1/3}(1 - \beta_1 n_j \sin\theta).$$

6. In conclusion, let us consider the case where

$$n_j\omega_s r_0 \sin\theta/c \ll 1. \qquad (23)$$

It is easy to see that in this case the greatest intensity occurs for the zeroth and first harmonics. For the zeroth harmonic $(s = 0)$, the expression for the radiated energy, when we satisfy the inequality (23), is identical with the Cerenkov term (13). In this connection we note that the picture of the radiation of an electron, moving along a vortex line in a medium with $n > 1$, as the radiation from a pair of perpendicular oscillating dipoles is incorrect (since it does not give the Cerenkov effect).

We now determine the energy corresponding to the first harmonic $(s = 1)$. To do this, we substitute in (14) the first terms of the expansion of the Bessel function and its derivative. We then find that

$$W_{1j} = \frac{Te^2\omega_1^2 n_j d\Omega\,[v_1(1+\alpha_j) + \beta\omega_1 n_j r_0 \beta_2 \sin\theta]^2}{16\pi c^3\,[1 - \beta_2\cos\theta\,(n_j + \omega_1 \partial n_j(\omega_1)/\partial\omega)]}. \qquad (24)$$

From this formula we see that only for motion of the electron in an isotropic non-dispersive dielectric $(n(\theta, \omega) = \text{const},\ \alpha_j = \pm\cos\theta;\ \beta_j = \pm\sin\theta)$ do we get the familiar expression for the energy radiated by a pair of mutually perpendicular dipoles oscillating with a phase shift of $\pi/2$ (cf., for example, Refs. 11, 12):

$$W_1 = \sum_{j=1}^{2} W_{1j} = \frac{Te^2p_0^2 n d\Omega}{8\pi c^3\,|1 - \beta_2 n\cos\theta|^5}\{(1-\beta_2 n\cos\theta)^2 + (\cos\theta - \beta_2 n)^2\},$$

where $p_0^2 = e^2r_0^2$ is the dipole moment.

If the electron moves in a circle, (24) becomes the following simple expression:

$$W_{1j} = (Te^2\Omega_0^2\beta_1^2 n_j d\Omega/16\pi c)(1 + \alpha_j)^2. \qquad (25)$$

From (25) it is easy to determine the ratio of the energy radiated in the extraordinary wave to the energy of the ordinary wave:

$$W_{11}/W_{12} = n_1(1+\alpha_1)^2/n_2(1+\alpha_2)^2.$$

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APPLICATION OF QUANTUM FIELD THEORY METHODS TO THE MANY BODY PROBLEM

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It is shown that the energy and damping of quasiparticles are determined by the poles of a single particle propagation function. The relation between the two-particle Green's function and the kinetic equation is established.

INTRODUCTION

In many cases, weakly-excited states of a system of interacting particles can be described approximately as an aggregate of elementary excitations — quasiparticles. In such a treatment the excited state of the system is described by fewer parameters than are needed for an exact description. Thus the elementary excitation is not a stationary state, but is rather a packet of stationary states with a narrow energy spread. The washing out of the packet leads to damping of the excitation. A description of the states of a system in terms of elementary excitations is possible if the energy spread of the packet, which determines its damping, is small compared to the excitation energy.

We shall consider the case of a homogeneous unbounded system. In such a system, the momentum operator commutes with the Hamiltonian, so that the excited states are characterised by the value of the momentum of the system, in addition to the other parameters.

Apparently, in all Fermi systems there are excitations analogous to the excitations in an ideal Fermi gas. The energy of such an excitation is $E(p_1, p_2) = \epsilon(p_1) - \epsilon(p_2)$, where $p_1, \epsilon(p_1)$, and $p_2, \epsilon(p_2)$ are the momenta and energies of the particle and hole which constitute the excitation. Here $p_1 > p_0 > p_2$, and p_0 is the limiting Fermi momentum for the quasiparticles.

A quasiparticle with momentum p, near to p_0, can reduce its energy, transferring another quasiparticle from the Fermi sphere to a state with $p' > p_0$. From the limitations imposed by the Pauli principle and the laws of conservation of energy and momentum, it follows that the probability for such a process, which determines the damping of the quasiparticles, is proportional to $(p - p_0)^2$. Thus the description of excited states of a Fermi system by means of quasiparticles is the more exact the closer the momentum of the quasiparticles to p_0.

The properties of the excitations are conveniently studied by the methods of quantum field theory, by introducing the Green's function of the system. Then the single particle Green's function determines the energy and the damping of the quasiparticles. However there may be excitations in the system whose energy is not describable as a sum of energies of quasiparticles. The energy spectrum of such excitations can be found from the two-particle Green's function. The two-particle Green's function, as we shall show later, also enables us to determine the behavior of the system in a weak external field.

In addition to the Green's function of the particles, we can also introduce the propagation function for the interaction between the particles. For example, for the problem of electrons in a metal interacting with the lattice, this propagation function is the Green's function of the phonon. The phonon Green's function determines the energy and damping of excitations of the lattice.

SINGLE PARTICLE GREEN'S FUNCTION AND ENERGY SPECTRUM

1. The single particle Green's function is defined, as usual, by

$$G(\mathbf{r}_1 t_1,\ \mathbf{r}_2 t_2) = i\langle T\{e^{iHt_1}\psi(\mathbf{r}_1)\,e^{-iH(t_1-t_2)}\psi^+(\mathbf{r}_2)\,e^{-iHt_2}\}\rangle, \quad (1)$$

where $\psi(\mathbf{r}) = \sum_{\mathbf{p}} a_{\mathbf{p}} e^{i\mathbf{p}\mathbf{r}}$; the average is taken over the ground state function of the Hamiltonian system H.

The Green's function of the field φ which provides the interaction between the particles is defined similarly:

$$D(\mathbf{r}_1 t_1,\ \mathbf{r}_2 t_2) = i\langle P\{e^{iHt_1}\varphi(\mathbf{r}_1)\,e^{iH(t_1-t_2)}\,\varphi(\mathbf{r}_2)\,e^{-iHt_2}\}\rangle. \quad (2)$$

In the case of the phonon field,

$$\varphi(\mathbf{r}) = \sum_{\mathbf{q}} \alpha_{\mathbf{q}} (b_{\mathbf{q}} + b^{+}_{-\mathbf{q}}) e^{i\mathbf{q}\mathbf{r}},$$

where b_q and b_q^+ are the phonon annihilation and creation operators.

In the absence of external fields, the functions G and D depend only on $r = |\mathbf{r}_1 - \mathbf{r}_2|$ and $\tau = t_1 - t_2$. Expanding the functions $G(r, \tau)$ and $D(r, \tau)$ in Fourier integrals, we get

$$\begin{aligned} G(r,\tau) &= \int \frac{dpd\varepsilon}{(2\pi)^4} G(p, \varepsilon) e^{i(\mathbf{pr}-\varepsilon\tau)}, \\ D(r,\tau) &= \int \frac{dqd\omega}{(2\pi)^4} D(q, \omega) e^{i(\mathbf{qr}-\omega\tau)}, \end{aligned} \tag{3}$$

where $G(p, \epsilon)$ and $D(q, \omega)$ are the Green's functions in momentum representation.

In the absence of interactions between the particles, we easily find from (1), for Fermi systems,*

$$G_0(r,\tau) = i \int \frac{d\mathbf{p}}{(2\pi)^3} e^{i(\mathbf{pr}-\varepsilon_p^0\tau)} \begin{cases} 1 - n_p, & \tau > 0 \\ -n_p, & \tau < 0, \end{cases}$$

where $n_p = a_p^+ a_p$. Going over to momentum representation by means of formula (3), we get

$$\begin{aligned} &G_0(p, \varepsilon) = 1/(\varepsilon_p^0 - \varepsilon - i\Delta), \\ &\Delta \to \begin{cases} +0 & p > p_0 \\ -0 & p < p_0. \end{cases} \end{aligned} \tag{4}$$

Similarly, we find from (3),

$$D_0(q,\omega) = \alpha_q^2 \left\{ \frac{1}{\omega_q^0 - \omega - i\delta} + \frac{1}{\omega_q^0 + \omega - i\delta} \right\}, \quad \delta \to +0. \tag{5}$$

2. We now go on to the Fourier transforms with respect to $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. From (1), we find for

$$G(p,\tau) = \frac{1}{2\pi} \int d\varepsilon G(d, \varepsilon) e^{-i\varepsilon\tau}$$

the expression

$$G(p,\tau) = i \begin{cases} < a_p e^{-iH\tau} a_p^+ > e^{iE_0\tau} & \tau > 0 \\ -< a_p^+ e^{iH\tau} a_p > e^{-iE_0\tau} & \tau < 0. \end{cases} \tag{6}$$

If we express the operators which appear in $G(p, \tau)$ in the energy representation, we have

$$G(p,\tau) = \begin{cases} i \sum_s |(a_p^+)_{s0}|^2 \exp\{-i(E_s - E_0)\tau\} & \tau > 0, \\ -i \sum_s |(a_p)_{s0}|^2 \exp\{i(E_s - E_0)\tau\} & \tau < 0. \end{cases} \tag{7}$$

Since the operator a_p^+ increases the momentum of the system by the amount **p**, and the number of particles in the system by unity, the summation for $\tau > 0$ extends over all states with momentum **p** and particle number N + 1, if the number of particles in the ground state was N and the momentum was equal to zero. Similarly, the summation for $\tau < 0$ is taken over states with particle number N − 1 and momentum −**p**. We use the notation

$$\begin{aligned} &E_s(N+1) - E_0(N) \\ &= \varepsilon_s(N+1) + E_0(N+1) - E_0(N) = \varepsilon_s + \mu, \end{aligned}$$

where $\mu = E_s(N+1) - E_0(N)$ is the chemical potential. The excitation energy $\epsilon_s = E_s(N+1) - E_0(N+1)$ is, by definition, positive. Similarly

$$\begin{aligned} &E_s(N-1) - E_0(N) \\ &= \varepsilon_s(N-1) - E_0(N) + E_0(N-1) = \varepsilon_s' - \mu'. \end{aligned}$$

The quantities ϵ_s' and μ' are identical with ϵ_s and μ to terms of order 1/N. We introduce the functions

$$\begin{aligned} A(p,E)\,dE &= \sum_s |(a_p^+)_{s0}|^2, \quad E < \varepsilon_s < E + dE, \\ B(p,E)\,dE &= \sum_s |(a_p)_{s0}|^2, \quad E < \varepsilon_s < E + dE \end{aligned} \tag{8}$$

and carry out a Fourier transformation with respect to τ in (7). We get

$$G(p,\varepsilon) = \int_0^\infty dE \left\{ \frac{A(p,E)}{E - \varepsilon + \mu - i\delta} - \frac{B(p,E)}{E + \varepsilon - \mu - i\delta} \right\}. \tag{9}$$

Formula (9) is Lehmann's[1] expansion for the single particle Green's function of a system consisting of a finite number of fermions. Using it, we can obtain some relations between the real and imaginary parts of the function $G(p, \epsilon)$. In fact, from the equality

$$\frac{1}{E - \varepsilon + \mu - i\delta} = \mathrm{P} \frac{1}{E - \varepsilon + \mu} + i\pi\delta(E - \varepsilon + \mu)$$

if follows that

$$\operatorname{Im} G(p,\varepsilon) = \pi \begin{cases} A(p, \varepsilon - \mu) & \varepsilon > \mu \\ -B(p, \mu - \varepsilon) & \varepsilon < \mu, \end{cases} \tag{10}$$

i.e., the imaginary part of the Green's function changes sign at the point $\epsilon = \mu$. Using (9) and (10), it is easy to obtain the formula* which gives the relation between the real and imaginary parts of G:

$$\operatorname{Re} G(p,\varepsilon) = \frac{1}{\pi} \int_{-\infty}^{\infty} \mathrm{P} \frac{\operatorname{Im} G(p,\varepsilon')}{\varepsilon' - \varepsilon} d\varepsilon'. \tag{11}$$

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*We shall give the formulas for Fermi systems. As is easily seen, most of the results also are valid for the case of Bose particles.

*This formula was obtained by L. D. Landau.

We remark that in the case of a system of bosons the expression for $G(p, \epsilon)$ differs from (9) only in a change in sign of $B(p, \epsilon)$. Thus for bosons, unlike (10), the imaginary part of $G(p, \epsilon)$ is positive for all p and ϵ. Formula (11) remains valid for bosons.

Similar formulas can be gotten for $D(q, \omega)$:

$$D(q, \tau) = \begin{cases} i \sum\limits_s |(\varphi_{-q})_{s0}|^2 \exp\{-i(E_s - E_0)\tau\} & \tau > 0 \\ i \sum\limits_s |(\varphi_q)_{s0}|^2 \exp\{i(E_s - E_0)\tau\} & \tau < 0. \end{cases} \tag{12}$$

Here we have made use of the reality of the field $\varphi : \varphi_q^+ = \varphi_q$. Unlike (7), the number of particles, N, in the states of the sum (12) is equal to the number of particles in the ground state. We use the notation

$$\Phi(q, \omega')\, d\omega' = \sum_s |(\varphi_{-q})_{s0}|^2 = \sum_s |(\varphi_q)_{s0}|^2, \qquad \omega' < E_s - E_0 < \omega' + d\omega', \tag{13}$$

Then

$$D(q, \tau) = \int_0^\infty d\omega' \Phi(q, \omega')\, e^{-i\omega'|\tau|}. \tag{14}$$

Taking the Fourier transform of (14) with respect to τ, we get

$$D(q, \omega) = \int_0^\infty d\omega' \Phi(q, \omega') \left\{ \frac{1}{\omega' - \omega - i\delta} + \frac{1}{\omega' + \omega - i\delta} \right\}. \tag{15}$$

From (15) we have

$$\operatorname{Im} D(q, \omega) = \pi \Phi(q, |\omega|) > 0; \tag{16}$$

$$\operatorname{Re} D(q, \omega) = \frac{1}{\pi} \int_0^\infty d\omega' \operatorname{Im} D(q, \omega')\, \mathrm{P} \left\{ \frac{1}{\omega' - \omega} + \frac{1}{\omega' + \omega} \right\}. \tag{17}$$

3. Let us examine the properties of the Green's function in the complex ϵ plane. Replacing E by $-E$ in the second term of (9), we get

$$G(p, \varepsilon) = \int_C \frac{F(p, E)}{E - \varepsilon + \mu}\, dE. \tag{18}$$

The integration contour C is shown in Fig. 1. The expression on the right side of (18) is an integral of the Cauchy type. Functions defined by such integrals are known^2 to be analytic throughout the plane except for the points on the contour of integration. In our case the integration contour C divides the plane of the complex variable $\epsilon - \mu$ into two regions, and the integral (18) defines two different functions: $f_I(\epsilon - \mu)$ which is analytic in region I, and $f_{II}(\epsilon - \mu)$ which is analytic in region II. The Green's function $G(p, \epsilon)$, which is defined by the values of the integral (12) on the real axis, coincides with f_{II} for $\epsilon < \mu$, and with f_I for $\epsilon > \mu$. Thus $G(p, \epsilon)$ is not an analytic function of ϵ, but has a singularity at $\epsilon = \mu$.

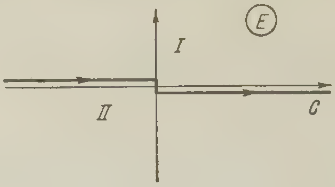

FIG. 1

Using the reality of the function $F(p, E)$, it is not difficult to show that the values of the integral (18) for points lying infinitely close to one another on opposite sides of the contour C are complex conjugates. Thus the function f_I, for negative values of $\epsilon - \mu$ lying above the contour C, is the complex conjugate of $G(p, \epsilon)$:

$$f_I(\varepsilon - \mu + 2i\delta) = G^*(p, \varepsilon), \quad \varepsilon < \mu. \tag{19}$$

Thus $G(p, \epsilon)$ for $\epsilon > \mu$, when continued analytically into the upper half-plane, coincides for $\epsilon < \mu$ with $G^*(p, \epsilon)$, or in other words, $G(p, \epsilon)$ for $\epsilon > \mu$ and $G^*(p, \epsilon)$ for $\epsilon < \mu$ comprise the analytic function f_I. Similarly, $G(p, \epsilon)$ for $\epsilon < \mu$, when continued analytically into the lower half-plane, coincides for $\epsilon > \mu$ with $G^*(p, \epsilon)$.

4. We shall now establish the connection of the single-particle Green's function with the spectrum of excitations. The Green's function $G(p, \tau)$ has a simple physical meaning. Suppose that initially the system is in the state $\Psi(0) = a_p^+ \Phi_0$, where Φ_0 is the ground state of the system of N particles (the physical "vacuum"). At the time $\tau > 0$, the wave function of the system is

$$\Psi(\tau) = e^{iH\tau} a_p^+ \Phi_0.$$

The function $G(p, \tau)$ is the probability for finding the system in state $\Psi(0)$ at time τ.

In fact,

$$(\Psi(0), \Psi(\tau)) = (\Phi_0 a_p e^{-iH\tau} a_p^+ \Phi_0) = -iG(p, \tau). \tag{20}$$

A similar relation holds for $\tau < 0$. According to (7) and (8), for $\tau > 0$

$$(\Psi(0), \Psi(\tau)) = e^{-i\mu\tau} \int_0^\infty A(p, E)\, e^{-iE\tau} dE. \tag{21}$$

In the presence of interaction, for values of p greater than p_0,

$$A(p, E) = \delta(E + \mu - \varepsilon_p^0) \text{ and } (\Psi(0), \Psi(\tau)) = e^{-i\varepsilon_p^0 \tau}.$$

When we switch on the interaction between particles, the δ function in $A(p, E)$ is replaced by a function having a sharp maximum near $E = \epsilon_p - \mu$, where ϵ_p is the energy of the quasiparticles.

Let us look at the behavior of the Green's function for large positive times. Suppose that the singularity of the analytic continuation of $A(p, E)$ into the lower half-plane, which is closest to the real axis is a simple pole at $E = \epsilon_p - \mu - i\Gamma$. Then, by shifting the contour of integration in (21) into the lower half-plane, we get

$$G(p, \tau) = ie^{-i\mu\tau}\int_C Ae^{-iE\tau}dE. \tag{21'}$$

The integration contour C is shown in Fig. 2. The non-exponential term in the function $G(p, \tau)$,

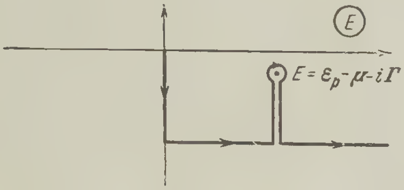

FIG. 2

which arises from the integration along the imaginary axis near $E = 0$, is of order $(\Gamma/\epsilon_p)^2$ for $\tau \gg 1/\Gamma$. Thus

$$G(p, \tau) = c_p e^{-i\varepsilon_p\tau - \Gamma\tau} + O[(\Gamma/\varepsilon_p)^2]. \tag{22}$$

This result can be interpreted in the following way: the state $\Psi(0)$ contains, with amplitude c_p, a packet describing a quasiparticle with energy ϵ_p and damping Γ. The values of ϵ_p and Γ are determined by the position of the pole of $A(p, E)$, i.e., by the imaginary part of $G(p, \epsilon)$ in the lower half-plane. The poles of $\operatorname{Im} G$ coincide with the poles of G or G^*, but the analytic continuation of the latter is $f_{II}(\epsilon - \mu)$, which is analytic in the lower half-plane. Thus the energy and damping of the excitations are determined by the real and imaginary parts of the poles of the analytic continuation of $G(p, \epsilon)$ for $\epsilon > \mu$ in the lower half-plane. Similarly, the energy and damping of holes in the Fermi distribution are given by the poles of the analytic continuation of $G(p, \epsilon)$ for $\epsilon < \mu$ into the upper half-plane.

Let us introduce the irreducible part of the proper energy of the particles, $\Sigma(p, \epsilon)$:

$$\begin{aligned} G^{-1}(p, \varepsilon) &= \varepsilon_p^0 - \varepsilon - \Sigma(p, \varepsilon) \\ &= \varepsilon_p^0 - \varepsilon - \Sigma_0(p, \varepsilon) - i\,\Sigma_1(p, \varepsilon), \end{aligned} \tag{23}$$

where Σ_0 and Σ_1 are the real and imaginary parts of Σ. The energy and damping of the quasiparticle are determined from the equation

$$\varepsilon_p^0 - (\varepsilon_p - i\Gamma) - \widetilde{\Sigma}(p, \varepsilon_p - i\Gamma) = 0, \tag{24}$$

where $\widetilde{\Sigma}(p, \epsilon)$ is the analytic continuation of $\Sigma(p, \epsilon)$ for $\epsilon > \mu$. For $\Gamma/\epsilon_p \ll 1$, we have approximately:

$$\begin{gathered} \varepsilon_p^0 - \varepsilon_p - \Sigma_0(p, \varepsilon_p) = 0, \\ \Gamma_p = \Sigma_1(p, \varepsilon_p) \Big/ \left[1 + \left(\frac{\partial \Sigma_0}{\partial \varepsilon}\right)_{\varepsilon = \varepsilon_p}\right]. \end{gathered} \tag{24'}$$

Analogous results hold for $D(q, \omega)$. Let $\Pi(q, \omega)$ be the irreducible part of the proper energy of the phonon

$$\begin{aligned} D(q, \omega) &= \alpha_q^2 / [\omega_q^{02} - \omega^2 - 2\omega_q^0\alpha_q^2\Pi(q, \omega)] \\ &= \alpha_q^2 / [\omega_q^{02} - \omega^2 - 2\omega_q^0\alpha_q^2\Pi_0(q, \omega) - 2i\omega_q^0\alpha_q^2\Pi_1(q, \omega)]. \end{aligned}$$

As above, the energy and damping of the phonon excitation are given by the poles of the analytic continuation of $D(q, \omega)$ for $\omega > 0$:

$$\omega_q^{02} - (\omega_q - i\gamma)^2 - 2\omega_q^0\alpha_q^2\widetilde{\Pi}(q, \omega_q - i\gamma) = 0, \tag{25}$$

where $\widetilde{\Pi}(q, \omega)$ is the analytic continuation of $\Pi(q, \omega)$ for $\omega > 0$. Approximately, for $\omega_q \gg \gamma$:

$$\begin{gathered} \omega_q^2 = \omega_q^{02} - 2\omega_q^0\alpha_q^2\Pi_0(q, \omega_q), \\ \gamma_q = \alpha_q^2\frac{\omega_q^0}{\omega_q}\Pi_1(q, \omega_q) \Big/ \left[1 + 2\omega_q^0\alpha_q^2\left(\frac{\partial \Pi_0}{\partial \omega}\right)_{\omega = \omega_q}\right]. \end{gathered} \tag{25'}$$

We determine the momentum p_0 from the condition $\mu = \epsilon_p$, where μ is the chemical potential of the system. Then, assuming that $\operatorname{Im} G$ is continuous at $\epsilon = \mu$, we find from (10) and (24′) that $\Gamma(p_0) = 0$. Thus the damping of the excitations goes to zero at the point p_0, which is determined by the equation $\epsilon_{p_0} = \mu$.

5. The single-particle Green's function also enables us to find other characteristics of the system. Thus the momentum distribution of the particles is related to the Green's function by[3]

$$n_p = i\int_C G(p, \varepsilon)\frac{d\varepsilon}{(2\pi)}, \tag{26}$$

where the contour C consists of the real axis and a semicircle of infinite radius in the upper half-plane.

As was shown in Ref. 3, when we pass through the point $p = p_0$, the pole of $G(p, \epsilon)$ which lies nearest to the real axis moves into the lower half of the ϵ plane, and is thus outside of the contour C of formula (26). Thus the jump in n_p at $p = p_0$ remains even in the presence of an arbitrary interaction.

Let us express the energy of the ground state of the system in terms of the function G. Differentiating $G(p, \tau)$ with respect to the time t, it is not difficult to obtain the formula

$$\left(i\frac{\partial}{\partial t} - \varepsilon_p^0\right)G(p, \tau) = -\delta(\tau) - i\langle T\{[H'(t), a_p(t)]\, a_p^+(t')\}\rangle, \tag{27}$$

where $H'(t)$ is the interaction Hamiltonian between particles, and $[H', a_p]$ is the commutator of the operators H' and a_p. Comparing (27) in the $(\mathbf{p}, \epsilon)$ representation with the expression relating G and G_0, we find the general form of the product $\Sigma(\mathbf{p}, \epsilon) G(\mathbf{p}, \epsilon)$:

$$\Sigma(p, \varepsilon) G(p, \varepsilon) = i \int d\tau e^{i\varepsilon\tau} \langle T\{[H'(t), a_{\mathbf{p}}(t)] a_p^{+}(t')\}\rangle. \tag{28}$$

By integrating (28) with the factor $e^{i\epsilon\Delta}$ and going to the limit $\Delta \rightarrow +0$, we get the formula

$$\lim_{\Delta\rightarrow +0} \int \frac{d\varepsilon}{2\pi} e^{i\varepsilon\Delta} \Sigma(p, \varepsilon) G(p, \varepsilon)$$

$$= \int_C \frac{d\varepsilon}{2\pi} \Sigma(p, \varepsilon) G(p, \varepsilon) = -i \langle a_{\mathbf{p}}^{+} [H', a_{\mathbf{p}}]\rangle,$$

or,

$$i \int_C \frac{d^4p}{(2\pi)^4} \Sigma(p, \varepsilon) G(p, \varepsilon) = \int \langle a_{\mathbf{p}}^{+} [H', a_{\mathbf{p}}]\rangle \frac{dp}{(2\pi)^3}, \tag{29}$$

where the contour C coincides with the integration contour in (26).

In the case of pair interaction between particles, the right side of (29) reduces to the average value of the interaction Hamiltonian

$$\langle H'\rangle = -\frac{i}{2} \int_C \frac{d^4p}{(2\pi)^4} \Sigma(p, \varepsilon) G(p, \varepsilon). \tag{30}$$

Adding the average value of the Hamiltonian of the non-interacting particles to (30), we get finally

$$E_0 = \frac{i}{(2\pi)^4} \int_C \left\{\varepsilon_p^0 - \frac{1}{2} \Sigma(p, \varepsilon)\right\} G(p, \varepsilon)\, d^4p. \tag{31}$$

Differentiation of E_0 with respect to the number of particles N gives the chemical potential, μ, of the system.

TWO-PARTICLE GREEN'S FUNCTION. KINETIC EQUATION

To study the energy spectrum and the behavior of the system in weak external fields, we must consider the two-particle Green's function. The two-particle Green's function $K(1, 2; 3, 4)$ is defined by

$$K(1,2;\ 3,4) = i \langle T\{\psi(1)\psi^{+}(2)\psi(3)\psi^{+}(4)\}\rangle, \tag{32}$$

where 1, 2; 3, 4 stand for the sets of coordinates of the space-time points. If $t_1, t_2 > t_3, t_4$, K can be written in the form

$$K(1,2;\ 3,4) = i \sum_s \chi_s(1,2) \tilde{\chi}_s(3,4), \tag{33}$$

where

$$\begin{aligned} \chi_s(1,2) &= (T\{\psi(1)\psi^{+}(2)\})_{0s}, \\ \chi_s(3,4) &= (\tilde{T}\{\psi(4)\psi^{+}(3)\})_{0s}^{*}; \end{aligned} \tag{34}$$

$\tilde{T}$ orders the operators in the reverse order from T. The functions $\chi_s(1, 2)$ for simultaneous times $t_1 = t_2$ have the physical meaning of wave functions describing the behavior of a particle and a hole in the state s. In the absence of external fields, the dependence on the coordinates of the "center of gravity" $X = (x_1 + x_2)/2$ can be separated off from χ:

$$\begin{gathered} \chi_s(x_1, x_2) = e^{ikX} f_{\mathbf{k},\omega}(x), \quad kX = \mathbf{k}\mathbf{R} - \omega T, \\ x = x_1 - x_2, \quad \mathbf{k} = \mathbf{p}_s - \mathbf{p}_0, \quad \omega = E_s - E_0, \end{gathered} \tag{35}$$

where $\mathbf{k}$ and ω are the momentum and energy of the excitation.

As will be shown later, the function $f_{\mathbf{k},\omega}(x)$ for $t_1 = t_2$ in momentum representation, i.e. $f_{\mathbf{k},\omega}(\mathbf{p})$, is the Fourier component of the distribution function $f(\mathbf{r}, \mathbf{p}, t)$. As a matter of fact, the density matrix, normalized to the total number of particles:

$$\begin{aligned} &P(\mathbf{r}, \mathbf{r}', t) \\ &= \sum_{i=1}^{N} \int \varphi^{*}(\mathbf{r}_1, \ldots, \mathbf{r}', \ldots, \mathbf{r}_N; t)\, \varphi(\mathbf{r}_1, \ldots, \mathbf{r}, \ldots, \mathbf{r}_N; t) \prod_{h\neq i} dV_h \end{aligned} \tag{36}$$

(where $\varphi(\mathbf{r}_1, \ldots \mathbf{r}_N; t)$ is the wave function of the system in configuration space), can be written as the average value of an integral operator with the kernel

$$(\hat{P}(\mathbf{r}, \mathbf{r}', t))_{\mathbf{r}_i', \mathbf{r}_i} = \sum_i \delta(\mathbf{r}' - \mathbf{r}_i')\, \delta(\mathbf{r} - \mathbf{r}_i), \tag{37}$$

which we may call the density matrix operator. In the occupation number representation, this operator has the form

$$\begin{aligned} &\hat{P}(\mathbf{r}, \mathbf{r}', t) \\ &= \int \psi^{+}(\mathbf{r}_1')\, \delta(\mathbf{r}' - \mathbf{r}_1')\, \delta(\mathbf{r} - \mathbf{r}_1)\, \psi(\mathbf{r}_1)\, dV_1 dV_1' = \psi^{+}(\mathbf{r}')\, \psi(\mathbf{r}). \end{aligned} \tag{37'}$$

$\chi_s(1, 2)$ for $t_1 = t_2$ is, to within a factor which is independent of points 1 and 2, the part of the density matrix which oscillates with frequency $\omega = E_s - E_0$. The Fourier component of the density matrix with respect to the coordinate $\mathbf{x} = \mathbf{r}_1 - \mathbf{r}_2$ is related, as we know, to the distribution function (the density matrix in mixed representation). Therefore the function f, to within a normalization factor, coincides with the Fourier component of the distribution function

$$f_{\mathbf{k},\omega}(\mathbf{p}) = c\int f(\mathbf{r}, \mathbf{p}, t)\, e^{-i(\mathbf{kr}-\omega t)}\, dvdt. \tag{38}$$

From (34) we see that the two-particle Green's function K is suitable for studying excited states of a system of N particles, in which there are particles and holes, while the single particle Green's function G enables us to investigate states of a system of N + 1 particles which differ from the ground state of the N particle system by the presence (or absence) of one quasiparticle. The essential feature of the states described by the two-particle Green's function K is the interaction between particle and hole. If this interaction leads only to scattering of the particle by the hole, then the energy of the excitation is equal to the energy of the particle and hole at infinity, $E = \epsilon(\mathbf{p}_1) - \epsilon(\mathbf{p}_2)$, $\mathbf{p} = \mathbf{p}_1 - \mathbf{p}_2$. In this case, the two-particle Green's function gives no new information concerning the energy spectrum of the system, beyond that from the single-particle function. In some cases the interaction can lead to the presence of excited states which can be interpreted as bound states of a particle and a hole. Such excited states were studied in the papers of Klimontovich and Silin[4,5] and Landau,[6] and were called zeroth sound.

As was shown in the papers of Schwinger,[7] and Gell-Mann and Low,[8] the equation for the function K has the form

$$K(x_1x_2;\ x_3x_4) = iG(x_1 - x_4)\,G(x_3 - x_2) - iG(x_1 - x_2)\,G(x_3 - x_4)$$
$$+ i\int G(x_1 - x_5)\,G(x_6 - x_2)\,\Gamma(x_5x_6,\ x_7x_8)\,K(x_7x_8;\ x_3x_4)\,d^4x_5d^4x_6d^4x_7d^4x_8, \tag{39}$$

Γ is a compact four-pole diagram, i.e., a set of graphs which start and end with a pair of solid lines, while the graphs cannot be split into parts which are joined only by a pair of solid lines. The free term of this equation describes the propagation of non-interacting particle and hole, and does not contain the frequencies corresponding to bound states. Therefore, extraction of the function χ_s, which describes bound states, leads, as was shown in Ref. 8, to the following homogeneous equation for χ:

$$\chi_s(x_1x_2) = i\int G(x_1-x_5)\,G(x_6-x_2)\,\Gamma(x_5x_6,\ x_7x_8)\,\chi_s(x_7x_8)\,d^4x_5d^4x_6d^4x_7d^4x_8. \tag{40}$$

Substituting χ_s in the form (35) into this equation, we get an eigenvalue problem whose solution gives us the frequency of zeroth sound and the functions χ_s. In the following, we shall limit ourselves to the case of a system of particles interacting with one another via a weak non-retarded potential V. To first order in the strength of the interaction, the zeroth order Green's functions should be used as the Green's function, while the compact four-pole Γ is given by the pair of graphs shown in Fig. 3. (The dotted lines on these graphs refer to the propagation function of the interaction, $-iV_q$. For our further work, we must include the spin of the particles. In the most usual case, of spin $\frac{1}{2}$, we can construct four functions χ_s:

$$\chi_s(1,2;\sigma_1,\sigma_2) = (T\{\psi_{\sigma_1}(1), \psi^+_{\sigma_2}(2)\})_{0s}. \tag{41}$$

It is not hard to see that the functions $\chi_s(1, 2;\ \frac{1}{2}, -\frac{1}{2})$ and $\chi_s(1, 2;\ -\frac{1}{2}, \frac{1}{2})$ correspond to excitations with spin 1 and projections +1 and −1, respectively. As the compact four-pole in the equation for these χ_s, we can use the Γ which is described by only the first of the graphs in Fig. 3; because the potential is independent of the spin

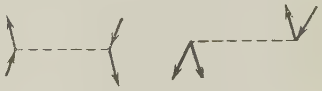

FIG. 3

variables, only a particle and hole with total spin equal to zero can participate in the second of the interactions. Thus the equation for these functions has the following form in momentum representation:

$$\chi_s(p_1, p_2;\sigma,\ -\sigma)$$
$$= iG_0(p_1)\,G_0(p_2)\int V_q\chi_s(p_1 + q, p_2 + q;\sigma, -\sigma)\,\frac{dq}{(2\pi)^4}\,. \tag{42}$$

In contrast to this case, the compact four-pole in the equation for the functions $\chi_s(1, 2;\ \frac{1}{2}, \frac{1}{2})$ and $\chi_s(1, 2;\ -\frac{1}{2}, -\frac{1}{2})$ is described by both of the graphs in Fig. 3. The equation for these functions is

$$\chi_s(p_1, p_2; \sigma, \sigma) = iG_0(p_1)\,G_0(p_2)\left\{\int V_q \chi_s(p_1+q, p_2+q; \sigma, \sigma)\frac{dq}{(2\pi)^4} - V_{p_1-p_2}\sum_{\sigma'}\int \chi_s\left(p+\frac{p_1-p_2}{2}, p-\frac{p_1-p_2}{2}; \sigma', \sigma'\right)\frac{dp}{(2\pi)^4}\right\}. \tag{43}$$

We introduce functions $\chi_S^+(p_1, p_2)$ and $\chi_S^-(p_1, p_2)$:

$$\chi_s^+(p_1, p_2) = \frac{1}{\sqrt{2}}\left\{\chi_s\left(p_1, p_2; \frac{1}{2}, \frac{1}{2}\right) + \chi_s\left(p_1, p_2; -\frac{1}{2}, -\frac{1}{2}\right)\right\}, \tag{44}$$

$$\chi_s^-(p_1, p_2) = \frac{1}{\sqrt{2}}\left\{\chi_s\left(p_1, p_2; \frac{1}{2}, \frac{1}{2}\right) - \chi_s\left(p_1, p_2; -\frac{1}{2}, -\frac{1}{2}\right)\right\}. \tag{44'}$$

The function $\chi_S^+(p_1, p_2)$ corresponds to excitation with spin zero, the function $\chi_S^-(p, p)$ to excitation with spin 1 and projection 0. The equations for these functions can be gotten by adding and subtracting Eq. (43) with spin values $\sigma = \frac{1}{2}$ and $\sigma = -\frac{1}{2}$. The equation for χ_S^- is the same as Eq. (42) for the function with spin 1 and projections ± 1. The equation for χ_S^+ is:

$$\chi_s^+(p_1, p_2) = iG_0(p_1)\,G_0(p_2)\left\{\int V_q \chi_s^+(p_1+q, p_2+q)\frac{dq}{(2\pi)^4} - 2V_{p_1-p_2}\int \chi_s^+\left(p+\frac{p_1-p_2}{2}, p-\frac{p_1-p_2}{2}\right)\frac{dp}{(2\pi)^4}\right\}. \tag{45}$$

Transforming to the "relative" momentum $\mathbf{k} = \mathbf{p}_1 = \mathbf{p}_2$, which is equal to the momentum of the excited state, and the "total" momentum $\mathbf{p} = (\mathbf{p}_1 + \mathbf{p}_2)/2$, we get the following equations for excitations with total spin zero and one, respectively:

$$\chi^0_{\mathbf{k}\omega}(p) = iG_0\left(p+\frac{k}{2}\right)G_0\left(p-\frac{k}{2}\right) \times \left\{\int V_q \chi^0_{\mathbf{k}\omega}(p+q)\frac{dq}{(2\pi)^4} - 2V_k\int \chi^0_{\mathbf{k}\omega}(p)\frac{dp}{(2\pi)^4}\right\}, \tag{46}$$

$$\chi^1_{\mathbf{k}\omega}(p) = iG_0\left(p+\frac{k}{2}\right)G_0\left(p-\frac{k}{2}\right)\int V_q \chi^1_{\mathbf{k}\omega}(p+q)\frac{dq}{(2\pi)^4}. \tag{47}$$

The function $\chi^0_{\mathbf{k}\omega}$ corresponds to excitation with total spin 0, the function $\chi'_{\mathbf{k}\omega}$ to excitation with total spin 1.

In the absence of retardation, the Fourier component of the potential does not depend on the fourth component of $q: V_q \equiv V(\mathbf{q})$, so that Eqs. (46) and (47) can be integrated with respect to ϵ (the fourth component of p). Integration of the function χ with respect to ϵ corresponds in coordinate representation to equating the times t_1 and t_2, so that as a result of the integration we get an equation for $f_{\mathbf{k}\omega}(\mathbf{p})$, the Fourier components of the distribution function:

$$f^0_{\mathbf{k}\omega}(\mathbf{p}) = \frac{n_0(\mathbf{p}+\mathbf{k}/2) - n_0(\mathbf{p}-\mathbf{k}/2)}{\omega - \mathbf{kp} - i\delta\,[n_0(\mathbf{p}+\mathbf{k}/2) - n_0(\mathbf{p}-\mathbf{k}/2)[}\left\{\int V(\mathbf{q})\,f^0_{\mathbf{k}\omega}(\mathbf{p}+\mathbf{q})\frac{d\mathbf{q}}{(2\pi)^3} - 2V(\mathbf{k})\int f^0_{\mathbf{k}\omega}(\mathbf{p})\frac{d\mathbf{p}}{(2\pi)^3}\right\}, \tag{46'}$$

$$f^1_{\mathbf{k}\omega}(\mathbf{p}) = \frac{n_0(\mathbf{p}+\mathbf{k}/2) - n_0(\mathbf{p}-\mathbf{k}/2)}{\omega - \mathbf{kp} - i\delta\,[n_0(\mathbf{p}+\mathbf{k}/2) - n_0(\mathbf{p}-\mathbf{k}/2)]}\int V(\mathbf{q})\,f^1_{\mathbf{k}\omega}(\mathbf{p}+\mathbf{q})\frac{d\mathbf{q}}{(2\pi)^3}, \tag{47'}$$

where $n_0(\mathbf{p})$ are the occupation numbers for noninteracting particles.

Let us consider the case of short range forces ($ap_0 \ll 1$, where a is the range of the potential). For excitations with low momentum k, the function $f_{\mathbf{k}\omega}(\mathbf{p})$ differs from zero over a narrow range of momenta near p_0. Because of this, the potential $V(\mathbf{q})$ can be taken out from under the integral sign and, like $V(\mathbf{k})$, replaced by $V(0)$. Writing the difference $n_0(\mathbf{p}+\mathbf{k}/2) - n_0(\mathbf{p}-\mathbf{k}/2)$ in the form $\frac{1}{2}\mathbf{k}\partial f_0/\partial\mathbf{p}$ (($f_0 = 2n_0(\mathbf{p})$ is the distribution function of the non-interacting particles in the ground state), we find

$$f^0_{\mathbf{k}\omega}(\mathbf{p}) = -\frac{1}{2}\frac{\mathbf{k}\,\partial f_0/\partial\mathbf{p}}{\omega - \mathbf{kp} - i\delta\,[n_0(\mathbf{p}+\mathbf{k}/2) - n_0(\mathbf{p}-\mathbf{k}/2)]}V(0)\int f^0_{\mathbf{k}\omega}(\mathbf{p})\,d\mathbf{p}/(2\pi)^3, \tag{46''}$$

$$f^1_{\mathbf{k}\omega}(\mathbf{p}) = \frac{1}{2}\frac{\mathbf{k}\,\partial f_0/\partial\mathbf{p}}{\omega - \mathbf{kp} - i\delta\,[n_0(\mathbf{p}+\mathbf{k}/2) - n_0(\mathbf{p}-\mathbf{k}/2)]}V(0)\int f^1_{\mathbf{k}\omega}(\mathbf{p})\,d\mathbf{p}/(2\pi)^3 \tag{47''}$$

Equation (46″) coincides with the kinetic equation in the self-consistent field approximation, but with the number of particles reduced by a factor of two. This equation has a solution only for $V(0) > 0$, i.e., in the case of repulsion between the particles.

The formal solution of equation (47″) for the case of attraction is not justified, because of the readjustment of the Fermi sphere caused by the formation of correlated pairs. Thus for a repulsive short-range potential, propagation of spinless zeroth sound is possible.

In the case of long-range repulsive forces, $V(\mathbf{k})$ has a pole for $k \to 0$, so that the second term in (46′) is much greater than the first, in which the integration makes the pole in V unimportant (we note that this result remains true in all approximations). Neglecting the first term, we write (46′) as

$$f^0_{\mathbf{k}\omega} = \frac{-\mathbf{k}\,\partial f_0/\partial \mathbf{p}}{\omega - \mathbf{kp} - i\delta\,[n_0(\mathbf{p}+\mathbf{k}/2) - n_0(\mathbf{p}-\mathbf{k}/2)]} \times V(\mathbf{k}) \int f^0_{\mathbf{k}\omega}(\mathbf{p}) \frac{d\mathbf{p}}{(2\pi)^3}. \tag{48}$$

This equation coincides with the kinetic equation for the $\mathbf{k}$, ω Fourier components of the distribution function in the self-consistent field approximation.

Let us treat the behavior of the system in a weak electromagnetic field $A(\mathbf{r}, t)$. The Hamiltonian for the interaction of the system with the field is

$$H' = \frac{1}{c} \int j^\alpha(\mathbf{r}, t)\, A^\alpha(\mathbf{r}, t)\, dv. \tag{49}$$

The summation over α extends from 1 to 4.

After the field is switched on at time t_0, the wave function of the system varies in time according to the law

$$\Phi(t) = T\left\{\exp\left(-i\int_{t_0}^{t} H' dt'\right)\right\}\Phi_0$$

$$= T\left\{\exp\left[-\frac{i}{c}\int_{t_0}^{t}\!\!\int j^\alpha(\mathbf{r}, t') A^\alpha(\mathbf{r}, t')\, dv dt'\right]\right\}\Phi_0, \tag{50}$$

or, in first approximation in powers of the external field,

$$\Phi(t) = \left\{1 - \frac{i}{c}\int_{t_0}^{t}\!\int j^\alpha(\mathbf{r}, t')\, A^\alpha(\mathbf{r}, t')\, dv dt'\right\}\Phi_0. \tag{50′}$$

In (50) and (50′) the current operator is taken in the Heisenberg representation for the unperturbed Hamiltonian of the system. The current of the system at time t is determined by the average value of the operator $j(\mathbf{r}, t)$ over the function $\Phi(t)$. Making use of the fact that the current of the system is equal to zero in the unperturbed state Φ_0, we easily find

$$j^\beta(\mathbf{r}, t) = -\frac{i}{c}\int_{t_0}^{t}\!\!\int dt' dv' \langle [j^\beta(\mathbf{r}, t), j^\alpha(\mathbf{r}', t')] \rangle A^\alpha(\mathbf{r}', t'), \tag{51}$$

where $<\ >$ denotes an average over the ground state of the system. For the $\mathbf{k}$-Fourier components, the relation (51) takes the form:

$$j^\beta_{\mathbf{k}}(t) = -\frac{i}{c}\int_{t}^{t} dt' \langle [j^\beta_{\mathbf{k}}(t), j^\alpha_{-\mathbf{k}}(t')] \rangle A^\alpha_{\mathbf{k}}(t')$$

$$= -\frac{i}{c}\int_{t_0}^{t} dt' \iint d\mathbf{p}\,d\mathbf{p}'\, j^\beta(\mathbf{p})\, j^\alpha(\mathbf{p}') \tag{51′}$$

$$\times \langle [a^+_{\mathbf{p}-\mathbf{k}/2}(t)\, a_{\mathbf{p}+\mathbf{k}/2}(t),\ a^+_{\mathbf{p}'+\mathbf{k}/2}(t')\, a_{\mathbf{p}'-\mathbf{k}/2}(t')] \rangle A^\alpha_{\mathbf{k}}(t'),$$

where $j^\alpha(\mathbf{p}) = p^\alpha$ for $\alpha = 1, 2, 3$, and $j^4(\mathbf{p}) = 1$. The average value of the commutator under the integral sign in (51) can be expressed in terms of the functions $f_{\mathbf{k}\omega}(\mathbf{p})$

$$\langle [a^+_{\mathbf{p}-\mathbf{k}/2}(t)\, a_{\mathbf{p}+\mathbf{k}/2}(t),\ a^+_{\mathbf{p}'+\mathbf{k}/2}(t')\, a_{\mathbf{p}'-\mathbf{k}/2}(t')] \rangle$$

$$= \sum_s \{e^{-i\omega_s(t-t')} f_{\mathbf{k},\omega_s}(\mathbf{p})\, f^*_{\mathbf{k},\omega_s}(\mathbf{p}') - e^{i\omega_s(t-t')} f_{-\mathbf{k},\omega_s}(\mathbf{p}')\, f^*_{-\mathbf{k},\omega_s}(\mathbf{p})\}. \tag{52}$$

Thus the knowledge of this system of functions is sufficient for determining the current of the system. On the other hand, this commutator can be expressed directly in terms of the two-particle Green's function K. Denoting by $\widetilde{K}$ the two-particle Green's function in momentum representation for $t_1 = t_2 = t$ and $t_3 = t_4 = t'$ $(t - t' = \tau)$:

$$K\left(\mathbf{p}+\frac{\mathbf{k}}{2},\ t, \mathbf{p}-\frac{\mathbf{k}}{2}, t;\ \mathbf{p}'-\frac{\mathbf{k}}{2}, t', \mathbf{p}'+\frac{\mathbf{k}}{2},\ t'\right) \equiv \widetilde{K}(\mathbf{p}, \mathbf{p}', \mathbf{k}; \tau), \tag{53}$$

we easily find

$$\langle [a^+_{\mathbf{p}-\mathbf{k}/2}(t)\, a_{\mathbf{p}+\mathbf{k}/2}(t),\ a^+_{\mathbf{p}'+\mathbf{k}/2}(t')\, a_{\mathbf{p}'-\mathbf{k}/2}(t')] \rangle = -i\widetilde{K}(\mathbf{p}, \mathbf{p}', \mathbf{k}; \tau) + i\widetilde{K}^*(\mathbf{p}, \mathbf{p}', -\mathbf{k}; \tau). \tag{54}$$

Substituting (54) in (51′), we have

$$j^\alpha_{\mathbf{k}}(t) = -\frac{1}{c}\int^{t} dt' \iint d\mathbf{p}\,d\mathbf{p}'\, j^\alpha(\mathbf{p})\, j^\beta(\mathbf{p}')\, \{\widetilde{K}(\mathbf{p}, \mathbf{p}', \mathbf{k}; \tau) - \widetilde{K}^*(\mathbf{p}, \mathbf{p}', -\mathbf{k}; \tau)\} A^\beta_{\mathbf{k}}(t'). \tag{55}$$

Going to the limit of $t_0 \to -\infty$, we get the relation between the time Fourier components of $j(t)$ and $A(t)$:

$$j^\alpha_{\mathbf{k},\omega} = \varkappa_{\alpha,\beta}(\mathbf{k}, \omega)\, A^\beta_{\mathbf{k},\omega},$$

$$\varkappa_{\alpha,\beta}(\mathbf{k}, \omega) = \frac{i}{2\pi c}\int d\mathbf{p}\,d\mathbf{p}' \frac{d\omega'}{\omega - \omega' + i\delta} \{\widetilde{K}(\mathbf{p}, \mathbf{p}'; \mathbf{k}, \omega') - \widetilde{K}^*(\mathbf{p}, \mathbf{p}'; -\mathbf{k}, -\omega')\}, \tag{56}$$

where $\widetilde{K}(\mathbf{p}, \mathbf{p}'; \mathbf{k}, \omega)$ is the time Fourier component of the function $\widetilde{K}(\mathbf{p}, \mathbf{p}'; \mathbf{k}, \tau)$.

In conclusion, the authors express their thanks to L. D. Landau and S. T. Beliaev for interesting discussions.

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THE ENERGY SPECTRUM OF A NON-IDEAL FERMI GAS

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We have evaluated the energy spectrum and ground state energy of a non-ideal Fermi gas with repulsive interactions, using an expansion in powers of the ratio of the range of the potential to the mean distance apart of the particles (gas approximation). We have obtained the first two terms of the expansion.

INTRODUCTION

It is well known that in many cases one can consider the excited states of a system of interacting Fermi particles as a gas of elementary excitations — quasiparticles. The energy of a quasiparticle is determined by its momentum in such a way that the energy of the excitation of the system ϵ_s is equal to $\epsilon(p_1) - \epsilon(p_2)$, where $p_1 > p_0 > p_2$ with p_0 the momentum at the Fermi surface. Such a spectrum is called a spectrum of the "Fermi type." A description of a system by means of the method of quasiparticles is exact only in the case of an ideal gas. If there are interactions between the particles, the excited states of the "Fermi type" do not represent the exact stationary states of the systems. This leads to the damping of the quasiparticles.

It was shown in Ref. 1 that it is convenient to apply the methods of quantum field theory to determine the energy spectrum of a system. The energy $\epsilon(p)$ and attenuation $\gamma(p)$ of the quasiparticles can be found as the poles of the analytical continuation of the single-particle Green function $G(p)$. In the present paper we shall apply the methods of quantum field theory to the problem of a non-ideal Fermi gas in which the interaction between the particles is short range $na^3 \ll 1$ (n is the density of the particles in the system and a the range of the potential), but not necessarily weak. We assume that the radially symmetrical potential $V(r)$ is positive and that the interaction between the particles is not retarded. We expand in powers of the parameter $p_0 f_0$, where f_0 is the real part of the scattering amplitude for small momenta. We shall find the energy spectrum of the system and the ground state energy up to quadratic terms in this parameter. Terms corresponding to higher powers than the cubic can not be expressed by means of two-particle parameters which makes it difficult to obtain them in a general form.* This fact was first remarked on in Ref. 2 in connection with the evaluation of the ground state energy.

1. SINGLE PARTICLE GREEN FUNCTION. THE METHOD OF GRAPHS

It is well known that the single particle Green function of a system is given by the equation

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*The author is obliged to E. M. Lifshitz for this comment.

$$iG(x-x') = \langle T\{\psi(x)\psi^+(x')S\}\rangle / S_{00}, \qquad (1)$$

where ψ and ψ^+ are taken in the interaction representation and where the averaging is performed over the ground state of the non-interacting particles. The S-matrix of the system is in our case given by the equation

$$S \equiv T\left\{\exp\left(-i\int H'dt\right)\right\} = T\left\{\exp\left[-\frac{i}{2}\int V(\mathbf{r}_1-\mathbf{r}_2)\psi^+(\mathbf{r}_1 t)\psi^+(\mathbf{r}_2 t)\psi(\mathbf{r}_2 t)\psi(\mathbf{r}_1 t)\,dv_1 dv_2 dt\right]\right\} \qquad (2)$$

(we choose our units in such a way that $\hbar = m = 1$). It is convenient to change this definition slightly by adding a dependence on a time t_2 to the dependence on the variable $\mathbf{r}_2$ and by introducing an extra integration over time,

$$S = T\left\{\exp\left[-\frac{i}{2}\int dx_1 dx_2 U(x_1-x_2)\right.\right.$$

$$\left.\left.\psi^+(x_1)\psi^+(x_2)\psi(x_2)\psi(x_1)\right]\right\}, \qquad (2')$$

where

$$U(x_1-x_2) = V(\mathbf{r}_1-\mathbf{r}_2)\,\delta(t_1-t_2). \qquad (3)$$

In (2′) the integration over the variables x_1 and x_2 is taken over the whole of the infinite four-dimensional space.

If we want to expand the S-matrix of (1) in powers of the interaction U, we must know the average value of the T product of the ψ operators. According to Wick's theorem this T product can be written in the form of a sum of normal products and different connections between operators. To apply the methods of quantum field theory it is necessary to put the average values of the normal products equal to zero. This condition will be fulfilled, if we take ψ in the form

$$\psi(\mathbf{r}) = u(\mathbf{r}) + v^+(\mathbf{r}); \quad u(\mathbf{r}) = V^{-1/2}\sum_{p>p_0} a_p e^{i\mathbf{pr}},$$

$$v^+(\mathbf{r}) = V^{-1/2}\sum_{p<p_0} a_p e^{i\mathbf{pr}} \qquad (4)$$

and if we define the normal products as those products in which all operators u and v are on the right, and all operators u^+ and v^+ on the left. In this representation the operators u and v play the role of annihilation operators and the operators u^+ and v^+ the role of creation operators for particles and holes. The average of the N products is equal to zero and the connection of two operators is equal to the Green function of non-interacting particles,

$$\dot\psi(x)\dot\psi^+(x') = iG_0(x-x') = \langle T\{\psi(x)\psi^+(x')\}\rangle. \qquad (5)$$

In the momentum representation the Green function of non-interacting particles is of the form

$$G_0^{-1}(p) \equiv G_0^{-1}(\mathbf{p}, \varepsilon) = \varepsilon - \varepsilon_p^0 + i\delta\theta(\mathbf{p}), \qquad (6)$$

where $\epsilon_{\mathbf{p}}^0 = \frac{1}{2}p^2$ and

$$\theta(\mathbf{p}) = 1 - 2n_{\mathbf{p}} = \begin{cases} 1 & |\mathbf{p}| > p_0 \\ -1 & |\mathbf{p}| < p_0 \end{cases},$$

with $n_{\mathbf{p}}$ is the occupation number of the non-interacting particles in the ground state. We can use the corresponding graphs to assign a definite arrangement of T products to an assembly of connections. The graphs consist of full drawn and dotted lines; each full drawn line corresponds to a particle propagation function $iG_0(x-x')$, or in momentum representation $iG_0(p)$, and each dotted line to an interaction, $iU(x_1-x_2)$ or in momentum representation $iU(q)$, and at each vertex the law of conservation of the four-dimensional momentum is obeyed, $p_1 - p_2 + q = 0$. Since the interaction does not enter into the normal products, we can restrict ourselves in calculating them to the connections of operators which enter in the same H′, that is, to simultaneous operators. To determine those connections we note that in the interaction H′ the operators ψ^+ are to the left of the operators ψ so that $G_0(p, \tau)$ for $\tau = 0$ must necessarily be taken to be $G_0(\mathbf{p}_1, -0) = in_{\mathbf{p}}$.

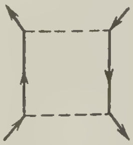

FIG. 1

For our further discussion it is very important that the interaction is not retarded [Eq. (3)]. It is convenient to indicate this absence of retardation by drawing the dotted lines in the graphs horizontally. This representation makes it possible to judge from the graphs the number of particles and holes taking part in the process. The process corresponding to the graph of Fig. 1, for instance, involves one particle and one hole since for each arrangement of vertices one of the lines of 1 or 2 is directed downwards on account of Eq. (3).

2. ESTIMATE OF THE GRAPHS. GAS APPROXIMATION

We introduce now the main part of the energy eigenvalue of the particle $\Sigma(p)$,

$$G^{-1}(p) = G_0^{-1}(p) - \Sigma(p) = \varepsilon - \varepsilon_p^0 - \Sigma(p) \,. \qquad (7)$$

In the first approximation of perturbation theory $\Sigma(p)$ is determined by the two graphs of Fig. 2. Graph 2a corresponds to a non-exchange scattering by a particle of the Fermi sea (background particle) and graph 2b to an exchange scattering. In the case in which we are interested where $p_0 a \ll 1$ both graphs are equal and are of the order $nV \sim p_0^2 \cdot p_0 a \cdot V_0 a^2$, where V is the Fourier component of the potential for small values of the momentum, and V_0 is the value of the potential inside its range.

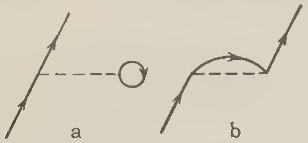

FIG. 2

We can consider now graphs which are more complicated in three different ways.

1. We can increase the number of dotted lines which connect solid lines which are in the same direction (graphs a and b in Fig. 3).

2. We can increase not only the dotted lines of the first kind, but also those which connect solid lines which are in opposite directions (graph c in Fig. 3).

3. We can increase the number of closed loops which are connected by dotted lines to the basic graph [it is well known that unconnected closed loops are eliminated by the denominator in Eq. (1)].

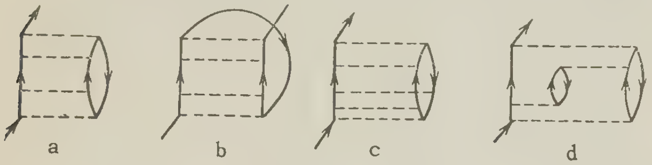

FIG. 3

Let us now estimate the value of these graphs. For graphs of the first kind each additional dotted line adds to Σ a factor $G_0^2 U(q)$ and one integration over the four-dimensional momentum q. As $U(q)$ only depends on $\mathbf{q}$ we get after integrating over the fourth component of q an integral of the form

$$\int \frac{dq}{q^2} V(\mathrm{q}), \qquad (8)$$

the convergence of which for large values of q is determined by the function $V(\mathbf{q})$. An estimate of this integral leads to Va^{-1} or $V_0 a^2$, i.e., the parameter of the perturbation theory. The collection of graphs of the first kind gives thus the perturbation theory series. For graphs of the second kind, the additional part has the form pictured in Fig. 1. Since one of the two lines corresponds to the occurrence of a hole, the momentum of which does not exceed the momentum at the Fermi surface, p_0, the integration over $\mathbf{q}$ in Eq. (8) will in that case be taken over a limited region of dimensions p_0. We obtain as a result $p_0 V$ or $p_0 a \cdot V_0 a^2$, i.e., graphs of the second kind contain apart from the perturbation theory parameter also an additional "gaseousness" parameter $p_0 a$. The difference between the magnitudes of graphs of the first and of the second kind can be given a simple physical interpretation. Indeed, graphs of the first kind correspond to a further approximation of the perturbation theory in terms of the interaction between two particles, while graphs of the second kind correspond to a further approximation in terms of an interaction between a particle and a hole. The interaction with a hole, however, is essentially an interaction with a background particle so that in the processes corresponding to the graphs 3a and 3b one background particle is taking place, but in the process 3c two background particles. This result can also be extended to graphs of the third kind (3d). Graphs 3c and 3d will be discarded. An exact estimate of these graphs which is not based upon perturbation theory is given in Sec. 5. The result of this estimate shows that graphs of the kind 3c and 3d can necessarily only be considered in the third approximation in terms of the small parameter $p_0 f$.

The graphs of the kind 3a and 3b which determine the energy eigenvalues of the particles in the first and second gas approximation can be given in the form of block diagrams as given in Fig. 4. The square denotes as usual the Feynman diagram assembly describing the interaction of two particles in the "ladder" approximation (Fig. 5). This quantity we shall denote by

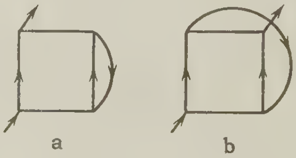

FIG. 4

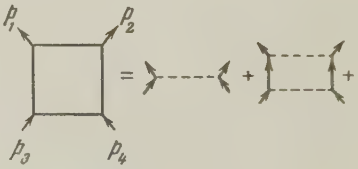

FIG. 5

$$-i\Gamma(p_1 p_2, \; p_3 p_4)$$

and we shall call it the effective interaction potential. Graphs 4a and 4b have the same structure as the graphs 2a and 2b corresponding to the first approximation of perturbation theory with one difference, however, namely that the dotted lines corresponding to the Born approximation have been taken into account in our effective potential. The values of the particle energy eigenvalues determined by these graphs is of the form

$$\Sigma(p) = -i\int dp' G_0(p')\,\Gamma(pp', pp') + i\int dp' G_0(p')\,\Gamma(pp', p'p). \tag{9}$$

Here and henceforth we shall use the following notation

$$dp = d\mathbf{p}\,d\varepsilon/2\pi, \quad d\mathbf{p} = dp_x dp_y dp_z/(2\pi)^3;$$
$$\delta(p) = 2\pi\delta(\mathbf{p})\,\delta(\varepsilon), \quad \delta(\mathbf{p}) = (2\pi)^3\,\delta(p_x)\,\delta(p_y)\,\delta(p_z).$$

Equation (9) does not take the presence of particle spin into account. To do that we note first of all that the Green function $G_0(p)$ contains the delta function $\delta_{s,s'}$ (where s and s' are the projections of the spin at the points x and x') which we have omitted, and this means that the particles propagate without change in spin. This result is also still valid in the case where the particles interact during their propagation with other particles, since the potential does not depend on the spin variable. The presence of spin leads thus to the appearance of a factor $\delta_{s_1 s_2}\delta_{s_3 s_4}$ in $\Gamma(p_1p_2, p_3p_4)$. The result of summing over the spin variable s' in Eq. (9) is a factor $2s+1$ for the first term and a factor 1 for the second one. This result has a simple physical meaning. The graph 4b corresponds to an exchange scattering in which one must take into account only those background particles which have a spin the projection of which coincides with the projection of the spin of the impinging particle. In contradistinction to this, graph 4a corresponds a non-exchange scattering process in which all particles in the Fermi sea must be taken into account. If we restrict ourselves to the case of particles of spin $\frac{1}{2}$ we get finally

$$\Sigma(p) = -2i\int dp' G_0(p')\,\Gamma(pp', pp') + i\int dp' G_0(p')\,\Gamma(pp', p'p). \tag{9'}$$

3. THE EFFECTIVE INTERACTION POTENTIAL

To determine the effective interaction potential we introduce a function Q which is connected with Γ by the following equation

$$\Gamma(p_1p_2, p_3p_4) = \int dq\,U(q)\,Q(p_1 - q, p_2 + q, p_3p_4). \tag{10}$$

The function Q differs from the two-particle Green function K in the "ladder" approximation by the absence of initial outside lines with momenta p_3 and p_4. We can thus obtain the equation for Q from the corresponding equation for the function K by dividing by the product $iG_0(p_3)\,iG_0(p_4)$. We have thus finally for $Q' = \delta(p_1 + p_2 - p_3 - p_4)\,Q$,

$$Q'(p_1p_2, p_3p_4) = \delta(p_1 - p_3)\,\delta(p_2 - p_4) + iG_0(p_1)\,G_0(p_2)\int dq\,U(q)\,Q'(p_1 - q, p_2 + q, p_3p_4). \tag{11}$$

If we go over to the relative momenta p and p' and the moment of the center of mass g,

$$p = (p_1 - p_2)/2, \quad p' = (p_3 - p_4)/2,$$
$$g = p_1 + p_2 = p_3 + p_4, \tag{12}$$

we get the equation

$$Q(p, p', g) \equiv Q\left(\frac{g}{2} + p,\ \frac{g}{2} - p;\ \frac{g}{2} + p',\ \frac{g}{2} - p'\right) = \delta(p - p') + iG_0\left(\frac{g}{2} + p\right)G_0\left(\frac{g}{2} - p\right)\int dq\,U(q)\,Q(p - q, p', g). \tag{13}$$

The potential $U(q)$ in Eq. (10) does not depend on the fourth component of q so that it is sufficient to know the function Q integrated over the fourth component of the relative momentum p, ϵ, in order to determine the effective potential. If we denote that function by $\chi(\mathbf{p}, \mathbf{p}', g)$ and integrate Eq. (13) over ϵ we find

$$\chi(\mathbf{p}, \mathbf{p}', g) - \frac{N(\mathbf{p})}{E - \mathbf{p}^2 + i\delta N(\mathbf{p})}\int d\mathbf{q}\,V(\mathbf{q})\,\chi(\mathbf{p} - \mathbf{q}, \mathbf{p}', g) = \delta(\mathbf{p} - \mathbf{p}'). \tag{14}$$

In Eq. (14) we have $E = g_0 - \frac{1}{4}g^2$ (where g_0 is the fourth component of g) and the factor $N(\mathbf{p})$ takes the Pauli exclusion principle into account inasfar as it applies to the initial background,

$$N(\mathbf{p}) = 1 - n_{g/2+\mathbf{p}} - n_{g/2-\mathbf{p}}. \tag{15}$$

For further calculations it is convenient to consider the effective potential as a function of the relative and total momenta which are defined by Eqs. (12). If we write

$$\Gamma(\mathbf{p}, \mathbf{p}', g) \equiv \Gamma\left(\frac{g}{2} + p,\ \frac{g}{2} - p;\ \frac{g}{2} + p',\ \frac{g}{2} - p'\right),$$

we obtain from (10) the following connection between the effective potential and the function χ,

$$\Gamma(\mathbf{p}, \mathbf{p}', g) = \int d\mathbf{q}\,V(\mathbf{q})\,\chi(\mathbf{p} - \mathbf{q}, \mathbf{p}', g). \tag{16}$$

Equation (14) can not be solved in its general form; our problem is to express the solution of this equation in terms of the scattering amplitudes of the particles. We consider first of all the problem of the scattering in vacuo. In that case $N(\mathbf{p}) \equiv 1$ and Eq. (14) is of the form

$$\chi_0(\mathbf{p},\mathbf{p}',g) - \frac{1}{E-\mathbf{p}^2+i\delta}\int d\mathbf{q}V(\mathbf{q})\chi_0(\mathbf{p},\mathbf{p}',g) = \delta(\mathbf{p}-\mathbf{p}'). \qquad (14')$$

Equation (14′), multiplied by $E - \mathbf{p}^2 + i\delta$, is the same as the inhomogeneous Schrödinger equation for the relative motion of two particles. It is therefore easy to express its solution in terms of $\psi_\mathbf{k}(\mathbf{p})$, the wave function of the relative motion of particles which are scattered ($\mathbf{k}$ is the relative momentum at infinity),

$$\chi_0(\mathbf{p},\mathbf{p}',g) = (E-p'^2+i\delta)\int d\mathbf{k}\frac{\psi_\mathbf{k}(\mathbf{p})\psi^*_\mathbf{k}(\mathbf{p}')}{E-k^2+i\delta}. \qquad (17)$$

Let us introduce the scattering amplitude $f(\mathbf{p},\mathbf{k})$ of the particles, by the equation

$$f(\mathbf{p},\mathbf{k}) = \int d\mathbf{q}V(\mathbf{q})\psi_\mathbf{k}(\mathbf{p}-\mathbf{q}). \qquad (18)$$

This amplitude differs from the usual one by a factor -4π, that is, the usual amplitude is equal to $-f(\mathbf{p},\mathbf{k})/4\pi$. The wave function $\psi_\mathbf{k}(\mathbf{p})$ is connected with the amplitude by the relation

$$\psi_\mathbf{k}(\mathbf{p}) = \delta(\mathbf{p}-\mathbf{k}) + \frac{f(\mathbf{p},\mathbf{k})}{k^2-p^2+i\delta}. \qquad (19)$$

Using this relation for $\psi^*_\mathbf{k}(\mathbf{p}')$ and substituting it into (17) we get

$$\chi_0(\mathbf{p},\mathbf{p}',g) = \psi_{\mathbf{p}'}(\mathbf{p}) + \int d\mathbf{k}\psi_\mathbf{k}(\mathbf{p})f^*(\mathbf{p}',\mathbf{k})\left\{\frac{1}{E-k^2+i\delta} + \frac{1}{k^2-p'^2-i\delta}\right\}, \qquad (17')$$

and we get for the effective interaction potential of particles in vacuo* which is defined by an equation which is analogous to equation (16)

$$\Gamma_0(\mathbf{p},\mathbf{p}',g) = f(\mathbf{p},\mathbf{p}') + \int d\mathbf{k}f(\mathbf{p},\mathbf{k})f^*(\mathbf{p}',\mathbf{k})\left\{\frac{1}{E-k^2+i\delta} + \frac{1}{k^2-p'^2-i\delta}\right\}. \qquad (20)$$

If we use Eq. (II) of the Appendix, we can express Γ_0 also in a slightly different, equivalent form,

$$\Gamma_0(\mathbf{p},\mathbf{p}',g) = f^*(\mathbf{p}',\mathbf{p}) + \int d\mathbf{k}f(\mathbf{p},\mathbf{k})f^*(\mathbf{p}',\mathbf{k})\left\{\frac{1}{E-k^2+i\delta} + \frac{1}{k^2-p^2+i\delta}\right\}. \qquad (20')$$

The effective interaction potential of particles in vacuo is thus equal in first approximation to the scattering amplitude $f(\mathbf{p},\mathbf{p}')$ or to $f(\mathbf{p},\mathbf{p}')$.

Going on to the solution of (14), we write it in the form

$$\chi(\mathbf{p},\mathbf{p}',g) - \frac{1}{E-p^2+i\delta}\int d\mathbf{q}V(\mathbf{q})\chi(\mathbf{p}-\mathbf{q},\mathbf{p}',\mathbf{q}) = \delta(\mathbf{p}-\mathbf{p}') + \left\{\frac{N(\mathbf{p})}{E-p^2+i\delta N(\mathbf{p})} - \frac{1}{E-p^2+i\delta}\right\}\Gamma(\mathbf{p},\mathbf{p}',g), \qquad (21)$$

where χ_0 is the Green function of the left hand side of this equation. Equation (21) admits thus of the following formal solution,

$$\chi(\mathbf{p},\mathbf{p}',g) = \chi_0(\mathbf{p},\mathbf{p}',g) + \int d\mathbf{k}\chi_0(\mathbf{p},\mathbf{k},g)\left\{\frac{N(\mathbf{k})}{E-k^2+i\delta N(\mathbf{k})} - \frac{1}{E-k^2+i\delta}\right\}\Gamma(\mathbf{k},\mathbf{p}',g).$$

If we apply (16) we can go over to an integral equation for the effective potential Γ,

$$\Gamma(\mathbf{p},\mathbf{p}',g) = \Gamma_0(\mathbf{p},\mathbf{p}',g) + \int d\mathbf{k}\Gamma_0(\mathbf{p},\mathbf{k},g)\left\{\frac{N(\mathbf{k})}{E-k^2+i\delta N(\mathbf{k})} - \frac{1}{E-k^2+i\delta}\right\}\Gamma(\mathbf{k},\mathbf{p}',g). \qquad (22)$$

Equation (22) can be solved by iteration methods since the term involving the integral is small. Indeed, the difference within the braces is different from zero only if $N \neq 1$, i.e., if the variation of k is of the order of p_0. To a first approximation we can substitute the scattering amplitude f for Γ_0. We then get, for $E \sim p_0^2$, the following order-of-magnitude estimate for the term involving the integral in (22):

$$p_0^3 f\cdot p_0^{-2}\Gamma = p_0 f\cdot\Gamma \ll \Gamma.$$

If we now take second-order terms $\ll \Gamma$ into account we can substitute in the integral in (22) $f(\mathbf{p},\mathbf{k})$ for $\Gamma_0(\mathbf{p},\mathbf{k},g)$ and $\Gamma_0(\mathbf{k},\mathbf{p}',g) \approx f(\mathbf{p}',\mathbf{k})$ for $\Gamma(\mathbf{k},\mathbf{p}',g)$. We get the following result,

$$\Gamma(\mathbf{p},\mathbf{p}',g) = \Gamma_0(\mathbf{p},\mathbf{p}',g) + \int d\mathbf{k}f(\mathbf{p},\mathbf{k})f^*(\mathbf{p}',\mathbf{k})\left\{\frac{N(\mathbf{k})}{E-k^2+i\delta N(\mathbf{k})} - \frac{1}{E-k^2+i\delta}\right\}$$
$$= f(\mathbf{p},\mathbf{p}') + \int d\mathbf{k}f(\mathbf{p},\mathbf{k})f^*(\mathbf{p}',\mathbf{k})\left\{\frac{N(\mathbf{k})}{E-k^2+i\delta N(\mathbf{k})} + \frac{1}{k^2-p'^2+i\delta}\right\}. \qquad (23)$$

As can be seen from Eq. (23), the effective interaction potential Γ, like Γ_0, is equal in first approximation to the scattering amplitude $f(\mathbf{p},\mathbf{p}')$. In the second part, the integral converges for values of k^2 of the order of the larger of the quantities E or p'^2. For the energy eigenvalues of the particles $\Sigma(\mathbf{p},\epsilon)$ in which we are interested, and for values of ϵ near $\frac{1}{2}p^2$, both quantities E and p'^2 are of the order of magnitude p_0^2. The second term in (23) is thus $p_0 f$ times smaller than the first one.

If the momenta $\mathbf{p}$ and $\mathbf{p}'$ have the same absolute magnitude we can write the effective potential in a slightly different form. Using the equation

$$\frac{1}{k^2-p'^2+i\delta} = \mathrm{P}\frac{1}{k^2-p^2} - i\pi\delta(k^2-p'^2)$$

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*The problems connected with Γ_0 were solved jointly with S. T. Beliaev, who was working simultaneously on the analogous problem of a Bose gas.

and Eq. (IV) of the Appendix, we get

$$\Gamma(\mathbf{p}, \mathbf{p}', g) = \operatorname{Re} f(\mathbf{p}, \mathbf{p}') + \int d\mathbf{k} f(\mathbf{p}, \mathbf{k}) f(\mathbf{p}', \mathbf{k})\, \mathrm{P} \frac{1}{k^2 - p'^2} + \int d\mathbf{k} f(\mathbf{p}, \mathbf{k}) f(\mathbf{p}', \mathbf{k}) \frac{N(\mathbf{k})}{E - k^2 + i\delta N(\mathbf{k})}, \qquad (23')$$

where the symbol P indicates the principal part of the integral.

4. THE ENERGY SPECTRUM OF THE SYSTEM. GROUND STATE ENERGY

The main part of the energy eigenvalues of the particles can be written in the form

$$\Sigma(p) = -2i \int \frac{d\varepsilon'}{2\pi} d\mathbf{p}' G_0(p') \Gamma(\mathbf{q}, \mathbf{q}, g) + i \int \frac{d\varepsilon'}{2\pi} d\mathbf{p}' G_0(p') \Gamma(\mathbf{q}, -\mathbf{q}, g),$$

where $\mathbf{q} = (\mathbf{p} - \mathbf{p}')/2$, $g = p + p'$. The first two terms for Γ in (23′) do not depend on the fourth component of the momentum so that integrating them over ϵ' reduces to calculating

$$\int \frac{d\varepsilon'}{2\pi} G_0(\mathbf{p}', \varepsilon').$$

This integral is equal to $G_0(\mathbf{p}', \tau)$ with $\tau = 0$, i.e., to $G_0(\mathbf{p}', -0) = in_{\mathbf{p}'}$, according to the considerations of Sec. 1. The integral over ϵ' of the last term of (23′) is elementary. We get the following results

$$\Sigma(p) = \Sigma_1(p) + \Sigma_2(p), \qquad (24)$$

$$\Sigma_1(p) = 2 \int d\mathbf{p}' n_{\mathbf{p}'} \operatorname{Re} f(\mathbf{q}, \mathbf{q}) - \int d\mathbf{p}' n_{\mathbf{p}'} \operatorname{Re} f(\mathbf{q}, -\mathbf{q}), \qquad (25)$$

$$\Sigma_2(p) = \int d\mathbf{p}' d\mathbf{k} \{2 |f(\mathbf{q}, \mathbf{k})|^2 - f(\mathbf{q}, \mathbf{k}) f^*(-\mathbf{q}, \mathbf{k})\} \times \Big\{ n_{\mathbf{p}'} \mathrm{P} \frac{1}{k^2 - q^2} - N(\mathbf{k}) \frac{N(\mathbf{k}) - \theta(\mathbf{p}')}{2} \frac{1}{k^2 - q^2 - \varepsilon + \varepsilon_p^0 + i\delta[\theta(\mathbf{p}') - N(\mathbf{k})]} \Big\}. \qquad (26)$$

For wavelengths that are considerably longer than the range a of the potential, the real part of the amplitude will not depend on the momenta. If we therefore restrict our considerations to excitations with momenta p which satisfy the condition

$$pf \ll 1, \qquad (27)$$

we can calculate the real part of the constant amplitude. Assuming that

$$\operatorname{Re} f = 4\pi f_0, \qquad (28)$$

where f_0 is the real part of the usual scattering amplitude with the opposite sign, and taking f_0 outside the integral sign in (25), we get for the first approximation to the energy eigenvalue

$$\Sigma_1 = 2\pi n f_0. \qquad (29)$$

To the first approximation in the gas parameter Σ is thus a real constant quantity. The Green function has in the first approximation the following form,

$$G_1(\mathbf{p}, \varepsilon) = 1/[\varepsilon - \varepsilon_p^0 - 2\pi n f_0 + i\delta\theta(\mathbf{p})]. \qquad (30)$$

If we introduce instead of ϵ a new variable ϵ',

$$\varepsilon' = \varepsilon - 2\pi n f_0, \qquad (31)$$

we can verify that $G_1(\mathbf{p}, \epsilon')$ is the same as the Green function of non-interacting particles. We can thus make our calculation more precise by assuming that everywhere instead of $G_0(\mathbf{p}, \epsilon)$ the Green function of the first approximation $G_1(\mathbf{p}, \epsilon)$ was used. If we make our considerations more precise in this way we must in all equations replace ϵ by ϵ' so that the Green function of the second approximation has the form

$$G_2^{-1}(\mathbf{p}, \varepsilon) = \varepsilon - \varepsilon_p^0 - 2\pi n f_0 - \Sigma_2(\mathbf{p}, \varepsilon') = \varepsilon' - \varepsilon_p^0 - \Sigma_2(\mathbf{p}, \varepsilon'), \qquad (32)$$

where $\Sigma_2(\mathbf{p}, \epsilon')$ is obtained from (26) by replacing ϵ by ϵ'.

We shall now calculate the energy eigenvalues of the second approximation, Σ_2. In the integral over the variable k in (26), the only appreciable contributions to the integral come from values of k in a region of the order of magnitude q. For excitations with momenta that satisfy condition (27), both arguments of the amplitude are small. The imaginary part of the amplitude can thus be neglected and the real part can be considered to be constant and can be taken outside the integral sign. Using the notation of Eq. (28), we get

$$\operatorname{Re} \Sigma_2 = 16\pi^2 f_0^2 \int d\mathbf{p}' d\mathbf{k} \Big\{ n_{\mathbf{p}'} \mathrm{P} \frac{1}{k^2 - q^2} - N(\mathbf{k}) \frac{N(\mathbf{k}) - \theta(\mathbf{p}')}{2} \mathrm{P} \frac{1}{k^2 - q^2 - \varepsilon' + \varepsilon_p^0} \Big\}, \qquad (26')$$

$$\operatorname{Im} \Sigma_2 = -4\pi^3 f_0^2 \int d\mathbf{p}' d\mathbf{k} N(\mathbf{k}) [N(\mathbf{k}) - \theta(\mathbf{p}')]^2 \delta(k^2 - q^2 - \varepsilon' + \varepsilon_p^0). \qquad (26'')$$

The energy spectrum of the system is determined by the poles of the analytical continuation of the Green function (1), or, in our approximation, by the equation

$$\varepsilon_p - \varepsilon_p^0 - 2\pi n f_0 - \Sigma_2(\mathbf{p}, \varepsilon_p^0) = 0. \qquad (33)$$

After some simple calculations we get for the energy ϵ_p and the damping γ of the quasiparticles the following expressions

$$\frac{\varepsilon_p}{p_0^2} = \frac{1}{2}x^2 + \frac{2}{3\pi}p_0f_0 + \frac{2}{15\pi^2}p_0^2f_0^2\frac{1}{x}\left\{11x + 2x^5\ln\frac{x^2}{|x^2-1|} - 10(x^2-1)\ln\left|\frac{x+1}{x-1}\right| - (2-x^2)^{5/2}\ln\left|\frac{1+x\sqrt{2-x^2}}{1-x\sqrt{2-x^2}}\right|\right\}, \quad x \leqslant \sqrt{2},$$

$$\frac{\varepsilon_p}{p_0^2} = \frac{1}{2}x^2 + \frac{2}{3\pi}p_0f + \frac{2}{15\pi^2}p_0^2f_0^2\frac{1}{x}\left\{11x + 2x^5\ln\frac{x^2}{x^2-1} - 10(x^2-1)\ln\left|\frac{x+1}{x-1}\right| - 2(x^2-2)^{5/2}\cot^{-1}\sqrt{x^2-2}\right\}, \quad x \geqslant \sqrt{2}, \tag{34}$$

$$\frac{\gamma}{p_0^2} = -\frac{1}{4\pi}p_0^2f_0^2(1-x^2)^2, \; x \leqslant 1; \quad \frac{\gamma}{p_0^2} = \frac{1}{15\pi}p_0^2f_0^2\frac{1}{x}\{5x^2-7+2(2-x^2)^{5/2}\}, \; 1 \leqslant x \leqslant \sqrt{2}.$$

$$\frac{\gamma}{p_0^2} = \frac{1}{15\pi}p_0^2f_0^2\frac{5x^2-7}{x}, \quad \sqrt{2} \leqslant x, \tag{35}$$

where $x = p/p_0$. The expansion (34) in a series in the momenta has for momenta near the Fermi surface the form

$$\frac{\varepsilon_p}{p_0^2} = \frac{1}{2}x^2 + \frac{2}{3\pi}p_0f_0 + \frac{2}{15\pi^2}p_0^2f_0^2(11 - 2\ln 2) - \frac{8}{15\pi^2}p_0^2f_0^2(7\ln 2 - 1)(x-1) + \ldots \tag{34'}$$

From Eq. (34′) we can obtain the effective mass, m^*, of particles on the Fermi surface,

$$\frac{m^*}{m} = 1 + \frac{8}{15\pi^2}(7\ln 2 - 1)p_0^2f_0^2 \tag{36}$$

and the chemical potential of the system which, as is well known[1] is equal to the energy of the quasiparticles for $p = p_0$,

$$\mu = \frac{1}{2}p_0^2\left\{1 + \frac{4}{3\pi}p_0f_0 + \frac{4}{15\pi^2}(11 - 2\ln 2)p_0^2f_0^2\right\}. \tag{37}$$

These expressions are the same as the ones obtained by Abrikosov and Khalatnikov.[3] Using the formula $\mu = (\partial E_0/\partial N)_V$, we find for the ground state energy of the system,[4]

$$\frac{E_0}{N} = \frac{3}{10}p_0^2\left\{1 + \frac{10}{9\pi}p_0f_0 + \frac{4}{21\pi^2}(11 - 2\ln 2)p_0^2f_0^2\right\}. \tag{38}$$

The damping of the quasiparticles γ for momenta p which are nearly equal to p_0 is of the form

$$\gamma = -\frac{1}{\pi}p_0^2f_0^2(p_0 - p)^2, \quad p < p_0,$$

$$\gamma = \frac{1}{\pi}p_0^2f_0^2(p - p_0)^2, \quad p > p_0, \tag{35'}$$

i.e., proportional to the square of the deviation from the Fermi surface.

Let us now consider excitations with large momenta,

$$p_0 \ll p \ll 1/f_0.$$

The quadratic correction term of the quasiparticle energy ϵ_p is small for such momenta. The imaginary part has the form

$$\gamma = \frac{1}{3\pi}p_0^3f_0^2p = \frac{1}{4}np\sigma.$$

If we use the connection between the imaginary part of the scattering amplitude and the total cross section σ we find

$$\varepsilon_p - i\gamma \sim \varepsilon_p^0 - 2\pi nf(p/2, p/2), \tag{39}$$

where f is the usual scattering amplitude.

The evaluation of Σ_2 as a function of the variables p and ϵ leads to a very cumbersome expression. The expansion of its real part for the case when $|x-1| \ll 1$ and $|y| \ll 1$, where

$$y = (\varepsilon' - \varepsilon_p^0)/p_0^2,$$

is of the form

$$\mathrm{Re}\,\Sigma_2(p, \varepsilon) = \frac{1}{\pi^2}p_0^2f_0^2\left\{\frac{2}{15}(11 - 2\ln 2) - \frac{8}{15}(7\ln 2 - 1)(x-1) - 4\ln 2\cdot y\right\}. \tag{40}$$

From Eq. (40) we can get the renormalization constant of the Green function Z which is connected, as is well known,[5] with the discontinuity of the momentum distribution function of the particles,

$$Z = n(p_0 - 0) - n(p_0 + 0) = 1 - 2p_0^2f_0^2\frac{1}{\pi^2}4\ln 2 \tag{41}$$

As far as the imaginary part of Σ_2 as a function of the variables p and ϵ is concerned, there are a number of regions in which $\mathrm{Im}\,\Sigma$ has a different analytical form. We shall show that this function tends to zero at the value $\epsilon' = p_0^2$, independent of the value of the momentum. We write Eq. (26″) in the following form

$$\mathrm{Im}\,\Sigma(p, \varepsilon) = -4\pi^3f_0^2\int np'dp_1dp_2\{n_{p'}(1 - n_{p_1})(1 - n_{p_2}) - (1 - n_{p'})n_{p_1}n_{p_2}\} \times \delta\left(\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 - \frac{1}{2}p'^2 - \varepsilon'\right)\delta(p_1 + p_2 - p' - p).$$

The first term within the braces determines the attenuation of the quasiparticles, and the second one the attenuation of the holes. For $\epsilon' = \frac{1}{2}p_0^2$ each of those terms tends to zero since from the first one it follows that $p'^2 < p_0^2$, $\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 > p_0^2$, and from the second one that $p'^2 > p_0^2$, $\frac{1}{2}p_1^2 + \frac{1}{2}p_2^2 < p_0^2$, both in violation of the equation $p_1^2 + p_2^2 = p'^2 + p_0^2$. According to the general theory of Green functions of many-body systems, the value for which the imaginary part of the energy eigenvalue tends to zero determines the chemical potential of the system μ

(Ref. 1). The equation

$$\operatorname{Im}\Sigma(\mathrm{p}, p_0^2/2) = 0$$

shows that the expression which we have found for $\operatorname{Im}\Sigma$ makes it possible to determine the chemical potential only to a first approximation. We can make the result more accurate by including in ϵ' besides $2\pi m f_0$ also the second-order energy correction on the Fermi surface. This method of calculation is used by Beliaev.[6]

We shall now discuss the connection between the results obtained by us and the general theory of a Fermi liquid developed by Landau.[7]

Owing to the monotonic dependence of the quasiparticle energy ϵ_p on the momentum, the quasiparticles fill the Fermi sea up to a limiting momentum p_0. The occupation numbers of the quasiparticles are thus the same as the occupation numbers n_p of the non-interacting particles. This equality is also maintained for states near the ground state. It is easily seen that expressions (25), (26), and (26′) for the extra energy of the particles are correct for any distribution of non-interacting particles. To see this it is sufficient to define the operators u and v^+ in Eq. (4) by

$$u = V^{-1/2}\sum_{\mathrm{p}}(1 - n_{\mathrm{p}})\, a_{\mathrm{p}} e^{i\mathrm{pr}}, \quad v^+ = V^{-1/2}\sum_{\mathrm{p}} n_{\mathrm{p}} a_{\mathrm{p}} e^{i\mathrm{pr}}, \qquad (4')$$

where n_p is the occupation number of the non-interacting particles. In the particular case of the ground state, Eq. (4′) is the same as Eq. (4). The sum of expressions (25) and (26′) at $\epsilon' = \epsilon_p^0$ gives the energy of the quasiparticles as a functional of the distribution function n_p of the quasiparticles. The variational derivative $\delta\epsilon_p/\delta n_{p'}$ determines the function $f_L(\mathbf{p}, \mathbf{p}')$ which was introduced by Landau (after a summation over the spins s and s′),

$$f_L(\mathrm{p}, \mathrm{p}') = 4\pi f_0 + 32\pi^2 f_0^2 \int dk\, n_{\mathrm{k}} \Big\{ \mathrm{P}\,\frac{1}{k^2 + \mathrm{pp}' - \mathrm{k}(\mathrm{p} + \mathrm{p}')} + \mathrm{P}\,\frac{1}{(\mathrm{p}' - \mathrm{k})(\mathrm{p}' - \mathrm{p})} - \mathrm{P}\,\frac{1}{(\mathrm{p} - \mathrm{k})(\mathrm{p}' - \mathrm{p})} \Big\}. \qquad (42)$$

Using Eq. (11) of Ref. 7 we obtain an expression for the effective mass of the particles at the Fermi surface which is the same as expression (36).

5. ESTIMATE OF THE GRAPHS OMITTED. HIGHER APPROXIMATIONS

When we estimated in Sec. 2 the importance of the omitted graphs, we considered an additional interaction with the background particles, corresponding to a single action of the potential V(q) (one additional dotted line). In the case where perturbation theory can not be used this estimate is incorrect. The Born approximation is the first term, which is large, of a series, the sum of which (the scattering amplitude) is small. To obtain a correct estimate it is necessary to sum the graphs corresponding to all orders of interaction of real gas particles, that is, all ladders of dotted lines connecting two full drawn lines going in the same direction. This sum is equal to the effective interaction potential Γ. To construct the graphs it is thus convenient to use the effective potential Γ (square), and not the potential U (dotted line).

Figure 6 illustrates a construction consisting of graphs omitted by us. To estimate the value of Σ_3 defined by those graphs it is sufficient to consider Γ in the first gas approximation, Γ ≈ f. We get then

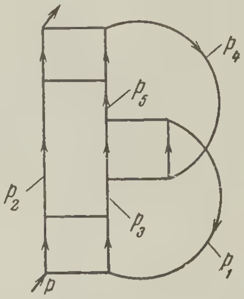

FIG. 6

$$\Sigma_3 = if^3 \int dp_1 dp_2 dp_4 G_0(p_1) G_0(p_2) G_0(p + p_1 - p_2) \times G_0(p_4) G_0(p + p_4 - p_2) \qquad (43)$$

It is important for us to show that the integral in (43) converges since in that case its value can only depend on the momentum p_0 (we assume that the momentum of the particles is in the neighborhood of p_0) and from dimensional considerations it follows that Σ_3 must be of the order $p_0^5 f^3$. After integrating over the fourth momentum component, there are only two energy denominators left in (43). If we then integrate over the momentum $\mathbf{p}_2$ we get a final quantity which is of the order of a reciprocal momentum. It is essential that the two remaining integrals over $\mathbf{p}_1$ and $\mathbf{p}_4$ be taken over a domain that is bounded by the Fermi surface. Indeed, in our approximation Γ does not depend on the fourth momentum component, that is, the interaction takes place instantaneously. The lines corresponding to the momenta p_1 and $p_3 = p + p_1 - p_2$ thus form a closed loop and one of these lines must correspond to the propagation of a hole (compare the graph of Fig. 1). Since the momentum of a hole is less than the Fermi momentum p_0, one or other of the two conditions, $p_1 < p_0$, or $|\mathbf{p} + \mathbf{p}_1 - \mathbf{p}_2| < p_0$, must be fulfilled. In each of these cases p_1

varies only in a bounded region. The same argument applied to the lines with the momenta p_4 and $p_5 = p + p_4 - p_2$ shows that the domain of integration of p_4 is bounded. We have thus proved the convergence of the integral, and Σ_3 is of the order of magnitude

$$\Sigma_3 \sim p_0^5 f^3 \sim p_0^2 n f^3. \qquad (44)$$

In conclusion the author wants to express his sincere thanks to A. B. Migdal and S. T. Beliaev for fruitful discussions and to A. F. Goriunov for help with the calculations.

APPENDIX

We shall derive certain relations to be satisfied by the scattering amplitude f. If we multiply Eq. (18) of the main text by $\psi_{\mathbf{k}}^*(\mathbf{p}')$ and integrate over k we find if we take into account that the ψ form a complete set of functions, that

$$V(\mathbf{p}-\mathbf{p}') = \int d\mathbf{k} f(\mathbf{p}, \mathbf{k}) \psi_{\mathbf{k}}^*(\mathbf{p}'),$$

or, if we substitute (19) for $\psi_{\mathbf{k}}^*(\mathbf{p}')$,

$$V(\mathbf{p}-\mathbf{p}') = f(\mathbf{p}, \mathbf{p}') + \int d\mathbf{k} \frac{f(\mathbf{p}, \mathbf{k}) f^*(\mathbf{p}', \mathbf{k})}{k^2 - p'^2 - i\delta}. \qquad (I)$$

If we use the condition that the potential be Hermitian we can obtain from (I) the following formula,

$$f(\mathbf{p}, \mathbf{p}') - f^*(\mathbf{p}', \mathbf{p}) = \int d\mathbf{k} f(\mathbf{p}, \mathbf{k}) f^*(\mathbf{p}', \mathbf{k}) \left\{ \frac{1}{k^2 - p^2 + i\delta} - \frac{1}{k^2 - p'^2 - i\delta} \right\}. \qquad (II)$$

In the case where the two vectors **p** and **p′** have the same absolute magnitude, the principal values of the integral cancel each other and (II) becomes

$$f(\mathbf{p}, \mathbf{p}') - f^*(\mathbf{p}', \mathbf{p}) = -2\pi i \int d\mathbf{k} f(\mathbf{p}, \mathbf{k}) f^*(\mathbf{p}', \mathbf{k}) \delta(p^2 - k^2). \qquad (III)$$

If we consider scattering in a central field of force, $f(\mathbf{p}, \mathbf{p}')$ can depend only on p^2 and $(\mathbf{p} \cdot \mathbf{p}')$, if $p = p'$, so that $f(\mathbf{p}, \mathbf{p}') = f(\mathbf{p}', \mathbf{p})$ and we get from (III)

$$\operatorname{Im} f(\mathbf{p}, \mathbf{p}') = -\frac{1}{16\pi^2} p \int d\mathbf{n} f(\mathbf{p}, p\mathbf{n}) f^*(\mathbf{p}', p\mathbf{n}) \qquad (IV)$$

where $|\mathbf{n}| = 1$. Equation (IV) contains as a special case the well known relation between the imaginary part of the scattering amplitude at sero angle and the total cross section.

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APPLICATION OF THE LOW INTEGRAL EQUATION METHOD TO THE PROBLEM OF PROTON-PROTON SCATTERING

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An integral equation of the Low type is derived and investigated in relation to the proton-proton scattering problem. The connection with the phenomenological quantum-theoretical treatment of proton-proton scattering is established. It is shown that the total scattering cross section observed in the medium-energy region can be obtained by solving the derived equation, taking into account only the most intense scattered wave $^1S_0$, provided one neglects in the equation the contribution from states that involve a number of real mesons and nucleon-antinucleon pairs in addition to two nucleons. Moreover, it must be assumed that the inhomogeneous term of the equation is computed in the nonrelativistic single-meson approximation.

CONTEMPORARY quantum field theory meets with serious difficulties in trying to explain even the most fundamental phenomena of nuclear physics. These difficulties are related to a great extent to the inapplicability of perturbation theory methods to the investigation of properties of interacting meson* and nucleon fields and to the fact that it thus becomes necessary to develop methods of calculation which are not derived from perturbation theory. Further development of such methods will enable us to find out to what extent does the present quantum field theory correspond to reality, and which of its features will remain in a future theory.

One of the methods not based on perturbation theory is the method of Low equations.†

Applying this method to a single nucleon problem, Chew and Low[3] found out that a whole class of phenomena taking place at low energies in the presence of one nucleon can be explained sufficiently well even with the help of the present theory.

It thus becomes interesting to try to apply the method of Low equations to a two-nucleon problem.

This work considers the problem of elastic nuclear proton-proton scattering from the point of view of the Low-type integral equation.

Section 1 of this work contains the derivation of the integral equation for proton-proton scattering in a form convenient for the subsequent calculations. States that contain in addition to two nucleons a certain number of real mesons and nucleon-antinucleon pairs are neglected. This is permissible in the region of nonrelativistic energies corresponding to the elastic proton-proton scattering. This equation contains, as an unknown function, a matrix element which is related on the energy shell to the proton-proton scattering amplitude. The inhomogeneous term of the equation is computed in the nonrelativistic single-meson approximation.

No perturbation theory, in the form of expansion in terms of the coupling constant of meson and nucleon fields, is used anywhere in the derivation of the integral equation, nor in its solution.

A very important circumstance is the fact that the obtained equation contains only renormalized quantities.

In Sec. 2 the integral equation is used to find the scattering cross-section assuming that the latter is determined by the $^1S_0$ phase.

In the integration with respect to the energy of the intermediate state, a cutoff limit is introduced equal to 5μ (μ is the rest mass of the meson).* It is important that the dependence of the scattering cross-section on the cutoff limit is very weak.

The expression obtained for the scattering cross-section agrees very well with the experimental data in the intermediate energy region from about 0.1 to 11 Mev.

1. INTEGRAL EQUATION FOR THE PROTON-PROTON SCATTERING

Consider the matrix element of the operator

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*Here and in the following the word "meson" means "π meson."

†Equations similar to those of Low also considered in Ref. 2.

*We let $c = \hbar = 1$.

$$S(\mathbf{k}'\nu') = a(\mathbf{k}'\nu', 0)\, H_I(0) - H_I(0)\, a(\mathbf{k}'\nu', 0)$$
$$= -\bar{u}^{\nu'}(\mathbf{k}')\left(\sum_{\lambda=1}^{4} \gamma_\lambda \frac{\partial}{\partial x_\lambda} + m\right)\psi(x)\Big|_{x=0},$$

taken between the functionals of the states of the system consisting of one and two protons:

$$(\Omega(\mathbf{q}'\mu'),\ S(\mathbf{k}'\nu')\,\Omega_{in}(\mathbf{k}\nu\mathbf{q}\mu)). \tag{1}$$

Here $\psi(x)$ is the nucleon field operator; $H_I(0)$ is the energy density of the interaction of meson and nucleon fields $H_I(x)$ (with the renormalization counterterms for $x = 0$ introduced); $\mathbf{k}$, $\mathbf{q}$ and $\mathbf{k}'$ $\mathbf{q}'$ are the space components of the proton four-momenta k, q and k', q', for which $\mathbf{k} + \mathbf{q} = \mathbf{k}' + \mathbf{q}'$ in the center of mass coordinate system, but in general $k_0 = q_0 \neq k_0' = q_0'$; ν, μ and ν', μ' are the spin variables; $u^{-\nu}(\mathbf{k}) = u^{\nu *}(\mathbf{k})\gamma_4$; $u^{\nu}(\mathbf{k})$ and γ_λ are the bispinors and Dirac matrices; m is the proton mass; and

$$a(\mathbf{k}\nu, 0) = a(\mathbf{k}\nu, x_0)\,|_{x_0=0} = \int d_3\mathbf{x} \exp(-i\mathbf{k}\mathbf{x})\, u^{\nu^*}(\mathbf{k})\, \psi(x)\,|_{x_0=0}.$$

The functional $\Omega_{in}(\mathbf{k}\mathbf{q})$ (we shall leave out the spin variables for brevity) can be represented in the form

$$\Omega_{in}(\mathbf{k}\mathbf{q}) = a_{in}^{+}(\mathbf{k})\,\Omega(\mathbf{q}). \tag{2}$$

In the above equation the operator $a_{in}^{+}(\mathbf{k})$ is the asymptotic form of the operator $\exp(-ik_0x_0) \times a^{+}(\mathbf{k}, x_0)$ [the operator $a^{+}(\mathbf{k}, x_0)$ is the Hermitian conjugate of the operator $a(\mathbf{k}, x_0)$]:

$$a_{in}^{+}(\mathbf{k}) = \exp(-ik_0x_0)\, a^{+}(\mathbf{k}, x_0) \text{ when } x_0 \to -\infty$$

and, by virtue of the equations of motion, it satisfies the relationship

$$a^{+}(\mathbf{k}, 0) = a_{in}^{+}(\mathbf{k})$$
$$+ i\int_{-\infty}^{0} dx_0 \int d_3\mathbf{x} \exp(ikx - iPx)\, S^{+}(\mathbf{k}) \exp(iPx), \tag{3}$$

where x_0 is the time; $kx = \mathbf{k}\mathbf{x} - k_0x_0$; $Px = \mathbf{P}\mathbf{x} - P_0x_0$; $(\mathbf{P}, P_0)$ – is the operator of the four-dimensional energy-momentum vector of the interacting meson and nucleon fields; the operator $S^{+}(\mathbf{k})$ is the Hermitian conjugate of $S(\mathbf{k})$.

For $k_0 = q_0 = k_0' = q_0'$, the matrix element (1) is related by a simple relationship to the proton-proton scattering amplitude $T(\mathbf{k}'\mathbf{q}', \mathbf{k}\mathbf{q})$ with initial proton quantum numbers $\mathbf{k}, \mathbf{q}$ and final quantum numbers $\mathbf{k}'$, $\mathbf{q}'$:

$$T(\mathbf{k}'\mathbf{q}', \mathbf{k}\mathbf{q}) = -\frac{k_0}{4\pi}\,(\Omega(\mathbf{q}'),\ S(\mathbf{k}')\,\Omega_{in}(\mathbf{k}\mathbf{q})). \tag{4}$$

Our problem is to derive an integral equation for the matrix element (1).

We shall start from the anti-commutator relationship for the nucleon field operators $\psi_\alpha(x)$ and $\psi_\beta^{+}(0)$, taken at the same instant of time $x_0 = 0$:

$$\{\psi_\alpha(x),\ \psi_\beta^{+}(0)\} = \delta_{\alpha\beta}\,\delta(x).$$

Multiplying both sides of the above equation by $u_\alpha^{*}(\mathbf{k}')\,u_\beta(\mathbf{k}) \exp(-i\mathbf{k}'\mathbf{x})$, integrating with respect to $\mathbf{x}$, and summing over α and β we obtain

$$\left\{\int d_3\mathbf{x} \exp(-i\mathbf{k}'\mathbf{x})\, u^{*}(\mathbf{k}')\,\psi(x),\ \psi^{+}(0)\, u(\mathbf{k})\right\} = u^{*}(\mathbf{k}')\, u(\mathbf{k}). \tag{5}$$

It follows from this that the operator expression on the left hand side of (5) is a number, and consequently its matrix element taken between the functionals $\Omega(\mathbf{q}')$ and $\Omega(\mathbf{q})$, which are mutually orthogonal for $\mathbf{q}' \neq \mathbf{q}$, is equal to zero:

$$(\Omega(\mathbf{q}'), \left\{\int d_3\mathbf{x} \exp(-i\mathbf{k}'\mathbf{x})\, u^{*}(\mathbf{k}')\,\psi(x),\right.$$
$$\left.\psi^{+}(0)\, u(\mathbf{k})\right\}\Omega(\mathbf{q})) = 0. \tag{6}$$

Introducing a complete orthonormal set of eigenvectors of the states Ω_{in}^{n} for the operator $(\mathbf{P}, P_0)$, and integrating with respect to the variable $\mathbf{x}$, we can rewrite Eq. (6) in the form:

$$\sum_{n(p^n=0)} (\Omega_{in}^{n}, \psi^{+}(0)\, u(\mathbf{k})\,\Omega(\mathbf{q}))\,(\Omega_{in}^{n}, \psi^{+}(0)\, u(\mathbf{k}')\,\Omega(\mathbf{q}'))^{*} + \sum_{n(p^n=\mathbf{q}-\mathbf{k}')} (\Omega(\mathbf{q}'), \psi^{+}(0)\, u(\mathbf{k})\,\Omega_{in}^{n})\,(\Omega(\mathbf{q}), \psi^{+}(0)\, u(\mathbf{k}')\,\Omega_{in}^{n})^{*} = 0, \tag{7}$$

where $\mathbf{p}^n$ is the momentum of the intermediate state described by the functional Ω_{in}^{n}.

Matrix elements of the operator $\psi^{+}(0)\,u(\mathbf{k})$, appearing in Eq. (7) (the matrix elements of the operator $\psi^{+}(0)\,u(\mathbf{k}')$ are obtained from the matrix elements of the operator $\psi^{+}(0)\,u(\mathbf{k})$ by replacing the quantum numbers $\mathbf{k}, \mathbf{q}$ by $\mathbf{k}', \mathbf{q}'$) can be transformed into the corresponding matrix elements of the operator $S^{+}(\mathbf{k})$ with the aid of the equations:

$$(\Omega_{in}^{n}, \psi^{+}(0)\, u(\mathbf{k})\,\Omega(\mathbf{q})) = \delta_{n,\mathbf{k}\mathbf{q}} - \frac{(\Omega_{in}^{n}, S^{+}(\mathbf{k})\,\Omega(\mathbf{q}))}{2k_0 - E_n + i\varepsilon} \quad (\mathbf{p}^n = 0),$$

$$(\Omega(\mathbf{q}'), \psi^{+}(0)\, u(\mathbf{k})\,\Omega_{in}^{n}) = -\frac{(\Omega(\mathbf{q}'), S^{+}(\mathbf{k})\,\Omega_{in}^{n})}{k_0 - q_0' + E_n} \quad (\mathbf{p}^n = \mathbf{q} - \mathbf{k}'), \tag{8}$$

where E_n is the energy of the intermediate state with momentum p^n; $\epsilon > 0$ and $\epsilon \to 0$; $\delta_{n,\mathbf{kq}} = 0$ if $\Omega_{in}^n \neq \Omega_{in}(\mathbf{kq})$, and $\delta_{n,\mathbf{kq}} = 1$ if $\Omega_{in}^n = \Omega_{in}(\mathbf{kq})$.

Equations (8) are easy to prove if we write the operator $\psi^+(0)$ on the left hand side of (8) in the form

$$\psi^+(0) = \sum_{\mathbf{p}} [u^*(\mathbf{p})\, a^+(\mathbf{p}, 0) + v^*(\mathbf{p})\, b(\mathbf{p}, 0)]$$

(the bispinor $v(\mathbf{p})$ and the operator $b(\mathbf{p}, 0)$ pertain to antinucleons) and then, following the calculational methods of Klein,^{4} we substitute for $a^+(\mathbf{p}, 0)$ and $b(\mathbf{p}, 0)$ their expressions from (3) and from an analogous formula for the antinucleon operator $b(\mathbf{p}, 0)$. Remembering relations (2) and taking into account the orthogonality and normalization properties of bispinors

$$u^{\nu^*}(\mathbf{k})\, u^{\mu}(\mathbf{k}) = \delta_{\nu\mu}, \quad v^{\nu^*}(-\mathbf{k})\, u^{\mu}(\mathbf{k}) = 0,$$

we arrive at Eq. (8).

Thus, substituting (8) into (7), we get

$$(k_0 - k_0')^{-1} [(\Omega(\mathbf{q}), S(\mathbf{k})\, \Omega_{in}(\mathbf{k}'\mathbf{q}'))^* - (\Omega(\mathbf{q}'), S(\mathbf{k}')\, \Omega_{in}(\mathbf{kq}))]$$

$$= 2 \sum_{n(\rho^n = 0)} \frac{(\Omega(\mathbf{q}'), S(\mathbf{k}')\, \Omega_{in}^n)\, (\Omega(\mathbf{q}), S(\mathbf{k})\, \Omega_{in}^n)^*}{(2k_0' - E_n - i\varepsilon)\, (2k_0 - E_n + i\varepsilon)} - V, \qquad (9)$$

where

$$V = - \sum_{n(\mathbf{p}^n = \mathbf{q} - \mathbf{k}')} \frac{(\Omega(\mathbf{q}'), S^+(\mathbf{k})\, \Omega_{in}^n)\, (\Omega(\mathbf{q}), S^+(\mathbf{k}')\, \Omega_{in}^n)^*}{E_n^2 - (q_0 - k_0')^2}.$$

In the first term of the right-hand side of Eq. (9) the summation is over all those intermediate states of the system consisting of two nucleons and of a certain arbitrary number $\ell \geq 0$ of real mesons and nucleon-antinucleon pairs, described by the functionals Ω_{in}^n, which are consistent with the laws of conservation of charge and momentum. In the expression for V the summation is over all those intermediate states of the system consisting of a certain arbitrary number $\ell \geq 0$ of real mesons and nucleon-antinucleon pairs, described by functionals Ω_{in}^n, which have a momentum $\mathbf{p}^n = \mathbf{q} - \mathbf{k}'$ and are consistent with the law of conservation of charge.

Identity (9) can be looked upon as the first of the infinite set of equations for the infinite number of matrix elements entering the first term of the right-hand side of identity (9). The quantity V determined by Eq. (10) plays the part of the inhomogeneous term in this equation. On the energy surface $E_n = 2k_0$ or $E_n = 2k_0'$, the desired matrix elements are related to the transition amplitudes for different scattering processes, leading to the formation of two protons, by simple relationships similar to (4).

If we use the nonrelativistic approximation, we can neglect in the first term of the right-hand side of (9) the states which, in addition to two nucleons, contain a certain number $\ell > 0$ of real mesons and nucleon-antinucleon pairs. In that case identity (9) can be considered as an equation for the matrix element (1), in which the inhomogeneous term V can be computed in the nonrelativistic single-meson approximation. Going over to the continuous spectrum and substituting integration for summation, we get

$$(k_0 - k_0')^{-1} [(\Omega(\mathbf{q}), S(\mathbf{k})\, \Omega_{in}(\mathbf{k}'\mathbf{q}'))^*$$

$$- (\Omega(\mathbf{q}'), S(\mathbf{k}')\, \Omega_{in}(\mathbf{kq}))] = \frac{1}{2(2\pi)^3} \times \qquad (11)$$

$$\sum \int \frac{(\Omega(\mathbf{q}'), S(\mathbf{k}')\, \Omega_{in}(\mathbf{ts}))\, (\Omega(\mathbf{q}), S(\mathbf{k})\, \Omega_{in}(\mathbf{ts}))^*\, \delta(\mathbf{t} + \mathbf{s})}{(k_0' - t_0 - i\varepsilon)\, (k_0 - t_0 + i\varepsilon)}\, d_3\mathbf{t}\, d_3\mathbf{s} - V,$$

where

$$V \sim g^2 \frac{(\bar{u}(\mathbf{q}')\, \gamma_5 u(\mathbf{k}))\, (\bar{u}(\mathbf{k}')\, \gamma_5 u(\mathbf{q}))}{[(\mathbf{q} - \mathbf{k}')^2 + \mu^2]^{3/2} + (q_0 - k_0')^2\, [(\mathbf{q} - \mathbf{k}')^2 + \mu^2]^{1/2}}. \qquad (12)$$

Here $\Omega_{in}(\mathbf{ts})$ is the functional of the intermediate state of the system consisting of two protons with momenta $\mathbf{t}$ and $\mathbf{s}$; the symbol Σ indicates summation over spins of protons in the intermediate state; the inhomogeneous term V is computed in the nonrelativistic single-meson approximation for the charge-symmetric meson theory with pseudo-scalar coupling; g^2 is the coupling constant of the meson and nucleon fields; $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4$; μ is the meson rest mass.

Equation (11), where the inhomogeneous term V is determined by formula (12), is the desired integral equation for the matrix element (1).

In order to establish a connection with the phenomenological theory of scattering from the potential U, it is convenient to use the following symbolic expression for the wave function $\psi_{\mathbf{kq}}(0)$ (at the time $t = 0$) of the system of two colliding protons with initial momenta $\mathbf{k}$ and $\mathbf{q}$:^{5}

$$\psi_{\mathbf{kq}}(0) = \varphi_{\mathbf{kq}} + \frac{1}{k_0 + q_0 - K - U + i\varepsilon}\, U \varphi_{\mathbf{kq}},$$

where K is the kinetic-energy operator for the system, and $\varphi_{\mathbf{kq}}$ is the eigenfunction of the operator K corresponding to the free motion of protons with momenta $\mathbf{k}$ and $\mathbf{q}$. We take the scalar product of this equation with $U\varphi_{\mathbf{k}'\mathbf{q}'}$. From the resultant equation we subtract its complex conjugate, in which the primed and unprimed quantum numbers have been transposed. If we now introduce a complete orthonormal set of functions $\psi_{\mathbf{ts}}(0)$, we obtain an equation analogous to (11) for the quantity

$$R(\mathbf{k}'\mathbf{q}', \mathbf{k}\mathbf{q}) = (\varphi_{\mathbf{k}'\mathbf{q}'}, U\psi_{\mathbf{k}\mathbf{q}}(0)),$$

which plays the part of the matrix element (1) (Ref. 5).

2. INVESTIGATION OF THE INTEGRAL EQUATION FOR THE $^1S_0$ WAVE COMPUTATION OF THE SCATTERING CROSS-SECTION

In this work we confine ourselves to dealing with the most intensive scattered wave $^1S_0$ for which the orbital angular momentum, the spin, and the total angular momentum are all equal to zero. We assume accordingly that the matrix element

$$(\Omega(\mathbf{q}'\mu'), S(\mathbf{k}'\nu')\Omega_{in}(\mathbf{k}\nu\mathbf{q}\mu)),$$

related to the proton-proton scattering amplitude by Eq. (4), is independent of the directions of momenta $\mathbf{k} = -\mathbf{q}$, $\mathbf{k}' = -\mathbf{q}'$ and is antisymmetric with respect to each pair of spin variables ν, μ and ν', μ'.

Introducing dimensionless energies

$$x_1 = m^{-1}(k_0 - m), \quad x_2 = m^{-1}(k_0' - m), \tag{13}$$

we let

$$\begin{gathered}(\Omega(\mathbf{q}'\mu'), S(\mathbf{k}'\nu')\Omega_{in}(\mathbf{k}\nu\mathbf{q}\mu)) = \\ -m^{-2}c(x_1)F(x_2x_1)\chi(\nu'\mu')\chi(\nu\mu),\end{gathered} \tag{14}$$

where $c(x_1) = m(kk_0)^{-1/2} \sim (2x_1)^{-1/4}$; the spin function $\chi(\nu\mu)$ is determined with the help of two relations $\chi(\nu\mu) = -\chi(\mu\nu)$ and $\chi(1-1) = 1$ (each of the spin variables ν, μ can take on the two values 1 and -1).

Substituting (14) into (11), summing over spin variables, and integrating over the directions of the momentum $\mathbf{t} = -\mathbf{s}$, we obtain an equation for the dimensionless function $F(x_2x_1)$:

$$\begin{gathered}(x_1 - x_2)^{-1}[c(x_1)F(x_2x_1) - c(x_2)F^*(x_1x_2)] \\ = \frac{1}{4\pi^2}\int_0^r \frac{F(x_2x')F^*(x_1x')}{(x' - x_1 - i\varepsilon)(x' - x_2 + i\varepsilon)}dx' - W(x_1x_2).\end{gathered} \tag{15}$$

Here $W(x_1x_2)$ is the isotropic part of the inhomogenous term V of (11), determined by (12), in units m^{-3}; $x' = m^{-1}(t_0 - m)$.

The range of integration over the dimensionless energy x' in the right hand side of (15) is from zero to the cutoff limit r, which is set equal to 0.19 (the kinetic energy in the laboratory system is then equal to 5μ, where μ is the meson rest mass). This value of the cutoff limit is the value chosen in meson-nucleon scattering problems.

Using a well-known relation

$$\frac{1}{x' - x \pm i\varepsilon} = \mathrm{P}\frac{1}{x' - x} \mp i\pi\delta(x' - x),$$

we rewrite equation (15) in the form

$$\begin{gathered}(x_1 - x_2)^{-1}\Big\{c(x_1)F(x_2x_1) - c(x_2)F^*(x_1x_2) \\ - \frac{i}{4\pi}[F(x_2x_2)F^*(x_1x_2) + F(x_2x_1)F^*(x_1x_1)]\Big\} \\ = \frac{1}{4\pi^2}\mathrm{P}\int_0^r \frac{F(x_2x')F^*(x_1x')}{(x' - x_1)(x' - x_2)}dx' - W(x_1x_2).\end{gathered} \tag{16}$$

At first we assume that $W(x_1x_2) = W(x_2x_1)$ is a product of functions depending only on one variable:

$$W(x_1x_2) = w(x_1)w(x_2).$$

We seek a solution of (16) in the form:

$$F(x_2x_1) = a_0(x_2) + a_1(x_2)(x_1 - x_2). \tag{17}$$

(Expression (17) for the function $F(x_2x_1)$ does not apply for $x_1 \to 0$, $x_2 \to 0$, since $W(x_1x_2)$ becomes infinite in the "non-physical region" of energies $x_1 < 0$, $x_2 < 0$, near $x_1 = x_2 = 0$.)

Substituting (17) into (16), carrying out the integration over the variable x', expanding both sides in powers of difference $x_1 - x_2$ for fixed x_2, and equating terms of the same powers of $x_1 - x_2$, we obtain the following equations for the desired functions $a(x)$ and $a_1(x)$:

$$c(a_0 - a_0^*) = \frac{i}{2\pi}a_0a_0^*; \tag{18}$$

$$\begin{gathered}2c'a_0 - 2ca_0^{*\prime} + 2ca_1 + 2ca_1^* - \frac{i}{\pi}a_0a_0^{*\prime} \\ + \frac{i}{2\pi}a_0a_1^* - \frac{i}{2\pi}a_0^*a_1 \\ = \frac{1}{2\pi^2}\left[(a_0a_1^* + a_0^*a_1)\ln\frac{r}{x} - \frac{1}{x}a_0a_0^* + ra_1a_1^*\right] - 2W(xx);\end{gathered} \tag{19}$$

$$\begin{gathered}c''a_0 - ca_0^{*\prime\prime} - \frac{i}{2\pi}a_0a_0^{*\prime\prime} + \frac{1}{2\pi^2x}a_0a_0^{*\prime} - \frac{1}{4\pi^2x^2}a_0a_0^* + W'(xx) \\ + a_1\left(2c' - \frac{i}{2\pi}a_0^{*\prime} + \frac{1}{2\pi^2x}a_0^* - \frac{1}{2\pi^2}a_0^{*\prime}\ln\frac{r}{x}\right) \\ + a_1^{*\prime}\left(2c + \frac{i}{2\pi}a_0 - \frac{1}{2\pi^2}a_0\ln\frac{r}{x}\right) - \frac{r}{2\pi^2}a_1a_1^{*\prime} = 0.\end{gathered} \tag{20}$$

Here $c = c(x)$, $a_0 = a_0(x)$, $a_1 = a_1(x)$; primes denote derivatives with respect to x of corresponding functions.

Equation (18) is obtained by letting the coefficients of $(x_1 - x_2)^{-1}$ in the left hand side of (16) be equal to zero. (Because of the relation

$$\lim_{x_1 \to x_2}\mathrm{P}\int_0^r \frac{dx'}{(x' - x_1)(x' - x_2)} = -\frac{r}{x_2(r - x_2)} \sim -\frac{1}{x_2}$$

the right-hand side of (16) does not contain such terms and has a finite limit for $x_1 \to x_2$.) Equation (18) thus shows that we can go to the limit $x_1 \to x_2$

in the left-hand side of (16), and is therefore the necessary consequence of the existence of a limit in the right hand side of (16) as $x_1 \to x_2$.

Equations (19) and (20) are obtained by equating the coefficients of the zeroth and first power of the difference $x_1 - x_2$ correspondingly in the left and right-hand sides of (16). (So far we do not consider coefficients of higher powers of the difference $x_1 - x_2$.)

Equation (18) and (19) have the property that each gives only one relation each for the real and imaginary parts of the functions contained in them. Thus, Eqs. (18), (19), and (20) give jointly four relations for the two complex functions $a_0(x)$ and $a_1(x)$.

Let us note here that Eq. (18) is the unitarity condition for the scattering matrix. This condition is therefore contained in the initial integral equation (16).

To solve the set of equations (18), (19), and (20), we start with Eq. (18) which leads to

$$a_0(x) = 4\pi c(x)[f(x) - i]^{-1}, \tag{21}$$

where $f(x)$ is a real function, so far arbitrary. We shall solve Eqs. (19) and (20) by successive approximations: first, we determine $f(x)$ from (19) assuming that $a_1(x) = 0$, then we put the obtained zeroth approximation for $f(x)$ into (20) to find $a_1(x)$ in the first approximation.

Direct substitution of (21) into (19), with the condition that $a_1(x) = 0$, leads to

$$f'(x) = -\frac{1}{\pi x} - (4\pi)^{-1}(2x)^{1/2}W(xx)[f^2(x) + 1]. \tag{22}$$

The quantity $W(xx)$ is the isotropic part of the inhomogeneous term V determined from Eq. (12) on the energy surface $k_0 = k_0'$ in units m^{-3}. Noting that in the non-relativistic approximation $(\bar{u}(p')\gamma_5\bar{u}(p)) \sim (2m)^{-1}x(\sigma, p' - p)$, where the vector σ is composed of the Pauli spin matrices $\sigma_1, \sigma_2, \sigma_3$, separating out the isotropic part of the term V, and introducing the dimensionless energy $x = m^{-1}(k_0 - m)$, we obtain approximately the following expression for the quantity $W(xx)$ for the value of the coupling constant $g^2 = 14.7$:

$$W(xx) \sim 0.8x(x + 0.006)^{-3/2}.$$

In our method the choice of the solution of Eq. (16) depends on the determination of the constant of integration in the solution of Eq. (22), which is an unknown function of the coupling constant. In principle this choice can be made by comparison with theory for low energies. However, we shall determine the constant of integration by requiring that the scattering cross-section for a certain energy coincide with the experimental value. For example, comparison with the proton-proton scattering data for the energy $x = 0.01$ leads to $f(0.01) \sim 0.9$. In this case the relative weight of the second term of the right hand side of equation (22) is ~ 0.003 compared to the first term; thus in the neighborhood of $x = 0.01$ the function $f(x)$ satisfies the following equation to a high degree of accuracy.

$$f'(x) = -\pi^{-1}x^{-1}. \tag{23}$$

Equation (23) will hold even better for $x < 0.01$, for as x gets smaller the relative weight of the second term of the right-hand side of eq. (22) will diminish only because $\pi^{-1}x^{-1}$ increases and $(2x)^{1/4}W(xx)$ decreases.

The solution of Eq. (23) has the form $f(x) = \pi^{-1}\ln(C/x)$ where the constant of integration C is determined from the condition $f(0.01) \sim 0.9$, and is approximately equal to the cutoff limit r. We thus obtain the following expression for the function $f(x)$:

$$f(x) = \frac{1}{\pi}\ln\frac{r}{x}. \tag{24}$$

Let us now consider the determination of $a_1(x)$ in the first approximation. Introducing

$$A \equiv 2c + \frac{i}{2\pi}a_0 - \frac{1}{2\pi^2}a_0 \ln\frac{r}{x},$$

let us rewrite Eq. (20) in the form

$$\frac{1}{2}a_0A^{*\prime\prime} - \frac{1}{2}a_0^{*\prime\prime}A + W'(xx) + a_1A^{*\prime} + a_1^{*\prime}A - \frac{r}{2\pi^2}a_1a_1^{*\prime} = 0.$$

Inserting here the zeroth approximation for $a_0(x)$ [Eqs. (21) and (24)] and noting that in this case $A = 0$, we get

$$W'(xx) - \frac{r}{2\pi^2}a_1a_1^{*\prime} = 0.$$

Since the function $W'(xx)$ is real, the solution of this equation has the form:

$$\begin{aligned} a_1(x) &= e^{i\alpha}b(x), \text{ when } b(x) = b^*(x), \\ b^2(x) &= 4\pi^2W(xx)/r + \beta. \end{aligned} \tag{25}$$

Here α and β are arbitrary real constants of integration. We shall see later that the constant β must be equal to zero and that the magnitude of the constant α is unimportant.

We obtained expressions (21), (22), and (25) for functions $a_0(x)$ and $a_1(x)$ by an approximate solution of the set of equations (18), (19), and (20). Actually, however, these expressions are the exact solutions of the set (18), (19), and (20) under the condition that $\beta = 0$. Moreover, the function $F(x_2x_1)$ determined from them with Eq. (17) is an exact solution of the integral equation (16) under the assumption that $W(x_1x_2) = w(x_1)w(x_2)$. (Of course, the equality of the coefficients of the first, second, third . . . powers of the difference $x_1 - x_2$ on the right and left hand side of Eq. (16) follow from this automatically. We did not consider these equalities before.)

Both the first and the second statement are easily proved by direct verification. Let us prove the first statement, for example. To do this it is sufficient to verify the following equalities, obtained from (19) by striking out terms containing only the unknown function $a(x)$, and also $a_0^*(x)$ and $a_0^{*\prime}(x)$:

$$2ca_1 + 2ca_1^* + \frac{i}{2\pi}a_0a_1^* - \frac{i}{2\pi}a_0^*a_1 = \frac{1}{2\pi^2}(a_0a_1^* + a_0^*a_1)\ln\frac{r}{x} + \frac{r}{2\pi^2}a_1a_1^* - 2W(xx).$$

The above becomes obvious if we rewrite it in the form

$$a_1A^* + a_1^*A = \frac{r}{2\pi^2}a_1a_1^* - 2W(xx)$$

and if we recall the relation (25) and that $A = 0$.

Thus, the function $F(x_2x_1)$ determined from Eqs. (17), (21), (24), and (25) is the exact solution of the integral equation (16), where $W(x_1x_2)$ has the form $W(x_1x_2) = w(x_1)w(x_2)$. If $W(x_1x_2) \neq w(x_1)w(x_2)$ then the function $F(x_1x_2)$ obviously does not satisfy Eq. (16) itself, but it satisfies an equation obtained from (16) by replacing $W(x_1x_2)$ by

$$[W(x_1x_1)W(x_2x_2)]^{1/2}.$$

Furthermore, insofar as for this value of $F(x_2x_1)$ [Eqs. (17), (21), (24), and (25)] the relative weight of the term $W(x_1x_2)$ in the right hand side of (16) is negligibly small compared to the first term on the right hand side of (16) (it is no bigger than 0.01 for $x_1 \leq 0.03$ and $x_2 \leq 0.03$ if the coupling constant is $g^2 = 14.7$), the function $F(x_2x_1)$ found by us is an approximate solution of the original integral equation (16) itself. In fact, when $x_1 \leq 0.03$ and $x_2 \leq 0.03$, this function $F(x_2x_1)$ coincides with $a_0(x_2)$ [when $a_1(x_2) = 0$, $F(x_2x_1) = a_0(x_2)$ is an exact solution of (16) for $W(x_1x_2) = 0$], since for these values of x_1 and x_2 the absolute magnitude of the term $a_1(x_2)(x_1 - x_2)$ is not more than 0.05 of the absolute magnitude of the quantity $a_0(x_2)$. Thus the approximate solution $F(x_2x_1)$ we obtained for (16) depends weakly on the difference $x_1 - x_2$, which characterizes the degree of removal from the energy surface $x_1 = x_2$.

Substituting the value obtained for $F(x_2x_1)$ into (14) and discarding the unimportant term $a_1(x_2) \times (x_1 - x_2)$, we arrive at the following expression for the sought matrix element (1):

$$(\Omega(\mathbf{q}'),\ S(\mathbf{k}')\,\Omega_{in}(\mathbf{k},\mathbf{q})) = \frac{4\pi}{(kk'k_0k_0')^{1/2}}\left(\frac{1}{\pi}\ln\frac{mr}{k_0'-m} - i\right)^{-1}.$$

Recalling Eq. (4), which relates the matrix element (1) to the proton-proton scattering amplitude on the energy surface $k_0 = k_0'$, we readily obtain an expression for the scattering cross-section of the ${}^1S_0$ wave:

$$d\sigma_{\text{c.m.}} = 8\cdot10^{-25}E^{-1}[1+\pi^{-2}\ln^2(E_mE^{-1})]\ \text{cm}^2\,\text{sterad}^{-1}.$$

Here E is the energy of the incident proton in the laboratory system, in Mev, and E_m is the cutoff energy, equal to 5μ, in Mev.

The expression obtained for the scattering cross-section depends weakly on the cutoff energy E_m and is in good agreement with experimental data in the intermediate energy region from about 0.1 to 100 Mev.

The method considered here for solving Eq. (16) does not enable us to determine the dependence of the matrix element (1), and of the scattering cross-section, on the coupling constant g^2. This is due to the fact that in the solution of Eq. (22)

$$f = f[x, C(g^2)] \equiv \varphi^{-1}(x, g^2),$$

we determined the integration constant $C = C(g^2)$, which is an unknown function of g^2, by comparison with experiment for only one value of g^2, namely $g_0^2 = 14.7$. The dependence of the constant of integration, and also of the scattering cross-section, on the coupling constant g^2 could in principle be determined by comparison with the theory for low energies. We shall make here only a few general remarks about this dependence.

Taking it into account that $\varphi(x, g^2) \to 0$ for $g^2 \to 0$, we expand $\varphi(x, g^2)$ in powers of $(g/g_0)^2$ and single out the first term of the expansion

$$\varphi(x, g^2) = (g/g_0)^2\varphi_0(x) + (g/g_0)^4\varphi_1(x, g^2),$$

where $\varphi_1(x, g^2)$ has a finite limit different from zero for $g^2 \to 0$, and where obviously

$$\varphi_0(x) + \varphi_1(x, g_0^2) = \pi\ln^{-1}(rx^{-1}).$$

When $g \sim g_0$, the most important term in this expansion is the second term, since, because of smallness of the quantity $W(xx)$ we have $\varphi_0(x) \ll \varphi_1(x, g_0^2)$. [This last inequality is directly related to those estimates which have enabled us to substitute Eq. (22) for (23).]

For small g^2, to the contrary, the principal term is the first term. For such values of g^2 the scattering cross-section determined by it behaves as the fourth power of the coupling constant, g^4.

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SPACE-TIME CORRELATION FUNCTIONS FOR A SYSTEM OF PARTICLES WITH ELECTROMAGNETIC INTERACTION

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A closed set of equations is obtained for the random functions $N_{\mathbf{qp}}(t)$. This set determines the number of particles at a given point $\mathbf{p}$, $\mathbf{q}$ in phase space at time t, and the vector and scalar potentials $\mathbf{A}$, φ. Chains of coupled equations for the moments of the random functions have been obtained by averaging from this set of equations. The equations are solved under the assumption that the random process in the system is stationary and uniform. Expressions are obtained which permit determination of space-time correlation functions of currents, densities, and vector potentials from a knowledge of simultaneous (equilibrium) correlation functions. Expressions are obtained for correlation functions of "extraneous" random electromagnetic fields and currents. In the absence of space dispersion these expressions become the familiar formulae derived by Leontovich and Rytov phenomenologically. An explicit expression is obtained for the complex dielectric constant of the medium

IN determining temporal correlation functions of random processes or space-time correlation functions for random fields we can introduce the concept of "extraneous" random forces or fields for which the correlation functions are known. Thus Rytov[1] has investigated fluctuations of the electromagnetic field by introducing extraneous emf's or extraneous fields, for which the correlation functions are assumed to be known. Similarly, the theory of hydrodynamical fluctuations is constructed by introducing "extraneous terms" with known correlation functions into the equation of motion of a fluid.[2]

It is the object of the present paper to obtain a closed set of approximate equations for the space-time correlation functions of a system of particles with electromagnetic interaction.

A similar problem for a classical system of particles with Coulomb interaction was considered by Tolmachev,[3] who used Bogoliubov's method to obtain a chain of equations for nonsimultaneous correlation functions. Through an approximate solution of this chain of equations Tolmachev[4] obtained an expression which relates the space-time correlation function of a system of charged particles with Coulomb interaction to the correlation function for $\tau = 0$.

In his investigation[5] of the spectra of elementary excitations in a system of centrally-interacting particles, the present author used the following equation for the random function:

$$N_{\mathbf{qp}}(t) = \sum_{i=1}^{N} \delta(\mathbf{q} - \mathbf{q}_i)\, \delta(\mathbf{p} - \mathbf{p}_i),$$

which defines the number of particles in the phase space region $d\mathbf{q}\,d\mathbf{p}$ around the point $\mathbf{q}$, $\mathbf{p}$ at time t. For the random function $N_{\mathbf{qp}}(t)$ there exists a multidimensional distribution function $F(\ldots N_{\mathbf{qp}}\ldots, t)$ for the probabilities of different values of $N_{\mathbf{qp}}$. The quantum analog of $N_{\mathbf{qp}}(t)$ is

$$N_{\mathbf{q}p}^{(Q)}(t) = (2\pi)^{-3}\int \Psi^{+}(\mathbf{q} - {}^{1}\!/_{2}\,\hbar\boldsymbol{\tau})\,\Psi(\mathbf{q} + {}^{1}\!/_{2}\,\hbar\boldsymbol{\tau})\,e^{-i\boldsymbol{\tau}\mathbf{r}}d\boldsymbol{\tau},$$

in which Ψ^{+} and Ψ are quantized wave functions which satisfy the familiar commutation relations.

In Ref. 6 the equations for $N_{\mathbf{qp}}(t)$ and $N_{\mathbf{qp}}^{Q}(t)$ were used to obtain chains of equations for nonsimultaneous distribution functions for systems of centrally-interacting particles. In the present paper the same method is used to investigate space-time correlation functions for a classical system of particles with electromagnetic interaction.

1. The Hamiltonian function of a classical system of N charged particles can be represented as follows:

$$\mathscr{H} = \sum_{i=1}^{N}\frac{1}{2m}\Big(\mathbf{P}_i - \frac{e}{c}\,\mathbf{A}(\mathbf{q}_i)\Big)^2 + e\sum_{i=1}^{N}\varphi(\mathbf{q}_i) +$$

$$+ \frac{1}{4\pi}\int \mathbf{E}\operatorname{grad}\varphi\, d\mathbf{q} + \frac{1}{8\pi}\int(\mathbf{E}^2 + \mathbf{H}^2)d\mathbf{q}, \qquad (1)$$

$$\mathbf{H} = \operatorname{curl}\mathbf{A}, \quad \mathbf{E} = -\operatorname{grad}\varphi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \quad \operatorname{div}\mathbf{A} = 0.$$

Here a Coulomb calibration has been used, by which the Coulomb interaction between particles can be distinguished immediately. It is assumed that electron charges are compensated by uniformly distributed positive ion charges.

If the vector potential is regarded as the field coordinate and the quantity

$$\boldsymbol{\Pi} = -\mathbf{E}/4\pi c = \mathbf{A}/4\pi c^2 + \operatorname{grad}\varphi/4\pi c$$

as the field momentum, one can introduce a distribution function

$$f(\mathbf{q}_1\ldots\mathbf{q}_N,\ \mathbf{P}_1\ldots\mathbf{P}_N, \ldots\mathbf{A}(\mathbf{q})\ldots, \ldots\boldsymbol{\Pi}(\mathbf{q})\ldots, t),$$

for the probabilities of different values of the coordinates and momenta of both the particles and the field. The equation for f can be obtained by means of a Hamiltonian equation corresponding to the Hamiltonian (1). The equilibrium solution of this equation is the Gibbs distribution.

The equation for f could be used to determine the space-time correlation functions. In the present work, however, we shall use a different method which is essentially as follows. We introduce the random function

$$N_{\mathbf{qP}}(t) = \sum_{i=1}^{N}\delta(\mathbf{q} - \mathbf{q}_i)\,\delta(\mathbf{P} - \mathbf{P}_i),$$

which defines the number of particles in the phase-space region $d\mathbf{q}\,d\mathbf{P}$ around the point $\mathbf{q}$, $\mathbf{P}$ at time t. By means of this definition we represent the Hamiltonian (1) in the equivalent form

$$\mathscr{H} = \int\frac{1}{2m}\Big(\mathbf{P} - \frac{e}{c}\,\mathbf{A}(\mathbf{q})\Big)^2 N_{\mathbf{qP}}d\mathbf{q}d\mathbf{P} + e\int\varphi(\mathbf{q})N_{\mathbf{qP}}d\mathbf{q}d\mathbf{P}$$

$$+ \frac{1}{4\pi}\int\mathbf{E}\operatorname{grad}\varphi\, d\mathbf{q} + \frac{1}{8\pi}\int(\mathbf{E}^2 + \mathbf{H}^2)\,d\mathbf{q}. \qquad (2)$$

The positively charged ionic background can be taken into account explicitly if $\int N_{\mathbf{qP}}d\mathbf{P}$ is replaced by $\int N_{\mathbf{qP}}d\mathbf{P} - n_{+}$, where n_{+} is the number of ions per unit volume.

Using the Hamiltonian (2), we obtain Hamilton's equations for the particles and variables which characterize the field:

$$\dot{\mathbf{q}} = \frac{\partial}{\partial\mathbf{P}}\frac{\delta\mathscr{H}}{\delta N_{\mathbf{qP}}} = \Big(\mathbf{P} - \frac{e}{c}\mathbf{A}\Big)/m, \qquad (3)$$

$$\dot{\mathbf{P}} = -\frac{\partial}{\partial\mathbf{q}}\frac{\delta H}{\delta N_{\mathbf{qP}}} =$$

$$-\frac{\partial}{\partial\mathbf{q}}\int\frac{e^2}{|\mathbf{q} - \mathbf{q}'|}N_{\mathbf{q}'\mathbf{P}'}\,d\mathbf{q}'d\mathbf{P}' - \frac{1}{2m}\frac{\partial}{\partial\mathbf{q}}\Big(\mathbf{P} - \frac{e}{c}\mathbf{A}\Big)^2, \qquad (4)$$

$$\dot{\mathbf{A}} = \delta\mathscr{H}/\delta\boldsymbol{\Pi} = 4\pi c^2\boldsymbol{\Pi} - c\operatorname{grad}\varphi, \qquad (5)$$

$$\dot{\boldsymbol{\Pi}} = -\frac{\delta H}{\delta\mathbf{A}} = \frac{1}{4\pi}\Delta\mathbf{A} + \frac{e}{cm}\int\Big(\mathbf{P} - \frac{e}{c}\mathbf{A}\Big)N_{\mathbf{qP}}\,d\mathbf{P}, \qquad (6)$$

$$\Delta\varphi = -4\pi e\int N_{\mathbf{qP}}\,d\mathbf{P}, \quad \operatorname{div}\mathbf{A} = 0. \qquad (7)$$

From Eqs. (4) and (5) and the continuity condition we derive an equation for the random function $N_{\mathbf{qp}}(t)$:

$$\frac{\partial N_{\mathbf{qP}}}{\partial t} + \frac{1}{m}\Big(\mathbf{P} - \frac{e}{c}\mathbf{A}\Big)\frac{\partial N_{\mathbf{qP}}}{\partial\mathbf{q}}$$

$$-\frac{\partial}{\partial\mathbf{q}}\int\frac{e^2}{|\mathbf{q} - \mathbf{q}'|}N_{\mathbf{q}'\mathbf{P}'}d\mathbf{q}'d\mathbf{P}'\cdot\frac{\partial N_{\mathbf{qP}}}{\partial\mathbf{P}} \qquad (8)$$

$$-\frac{1}{2m}\frac{\partial}{\partial\mathbf{q}}\Big(\mathbf{P} - \frac{e}{c}\mathbf{A}\Big)^2\frac{\partial N_{\mathbf{qP}}}{\partial\mathbf{P}} = 0.$$

In (5) – (8) we have a closed set of equations for the random functions $N_{\mathbf{qP}}$ and $\mathbf{A}$, $\boldsymbol{\Pi}$, φ.

It will be convenient hereinafter to use, instead of the canonical variables $\mathbf{q}$, $\mathbf{P}$, the variables $\mathbf{q}$, $\mathbf{p} = \mathbf{P} - e\mathbf{A}/c$ and the field variables $\mathbf{A}$, $\dot{\mathbf{A}} = 4\pi c^2\boldsymbol{\Pi} - c\operatorname{grad}\varphi$. Then $N_{\mathbf{qP}}d\mathbf{q}d\mathbf{P}$ becomes $N_{\mathbf{qp}}d\mathbf{q}d\mathbf{p}$, where $N_{\mathbf{qp}}(t)$ is the number of particles in a region $d\mathbf{q}\,d\mathbf{p}$ around a point $\mathbf{q}$, $\mathbf{p}$ of phase space at time t. After the transformation, Eqs. (5) – (8) become

$$\frac{\partial N_{\mathbf{qp}}}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial N_{\mathbf{qp}}}{\partial\mathbf{q}} - \Big\{e\frac{\partial\varphi}{\partial\mathbf{q}} + \frac{e}{c}\dot{\mathbf{A}} - \frac{e}{mc}\mathbf{p}\times\mathbf{H}\Big\}\frac{\partial N_{\mathbf{qp}}}{\partial\mathbf{p}} = 0; \qquad (9)$$

$$\Delta\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \frac{1}{c}\frac{\partial}{\partial t}\operatorname{grad}\varphi - \frac{4\pi e}{cm}\int\mathbf{p}'N_{\mathbf{qp}'}d\mathbf{p}'; \qquad (10)$$

$$\Delta\varphi = -4\pi e\int N_{\mathbf{qp}}\,d\mathbf{p}, \quad \operatorname{div}\mathbf{A} = 0. \tag{11}$$

The Hamiltonian function (2) in these variables has the simple form

$$\mathscr{H} = \int\frac{p^2}{2m}N_{\mathbf{qp}}\,d\mathbf{p}\,d\mathbf{q} + e\int\varphi(\mathbf{q})\,N_{\mathbf{qp}}\,d\mathbf{q}\,d\mathbf{p} - \frac{1}{8\pi}\int(\operatorname{grad}\varphi)^2\,d\mathbf{q}$$
$$+\frac{1}{8\pi}\int\left\{\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}\right)^2 + (\operatorname{curl}\mathbf{A})^2\right\}d\mathbf{q}. \tag{12}$$

Equations (9)—(11) correspond in external form to the set of self-consistent equations for the particle distribution function and electric and magnetic field strengths which was considered by Vlasov[7] and a number of other authors. Unlike the set of self-consistent equations, (9)—(11) are given for the random functions $N_{\mathbf{qp}}(t)$, $\mathbf{A}$, $\dot{\mathbf{A}}$ and in this case for a total description it is also necessary to know the distribution function

$$F(\ldots.N_{\mathbf{qp}}(t)\ldots,\mathbf{A}\ldots,\ldots\dot{\mathbf{A}}\ldots).$$

This procedure permits us to obtain from (9)—(11) a system of chains of equations for functions which determine average values of different combinations of $N_{\mathbf{qp}}(t)$ and $\mathbf{A}(\mathbf{q}, t)$ at different points of space and at different times. For this purpose, (9)—(11) must be averaged by means of F after they are multiplied by $N_{\mathbf{q}'\mathbf{p}'}(t')$, $\mathbf{A}(\mathbf{q}', t')$ or their combination. The first two equations of the chains are obtained by direct averaging of (9)—(11). Using a bar to denote averaging by means of F, we obtain

$$\frac{\partial\overline{N}_{\mathbf{qp}}}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial\overline{N}_{\mathbf{qp}}}{\partial\mathbf{q}} - \frac{\partial}{\partial\mathbf{q}}\int\frac{e^2}{|\mathbf{q}-\mathbf{q}'|}\overline{N_{\mathbf{q}'\mathbf{p}'}d\mathbf{q}'\,d\mathbf{p}'\,\frac{\partial N_{\mathbf{qp}}}{\partial\mathbf{p}}}$$
$$+\overline{\left(-\frac{e}{c}\frac{\partial\mathbf{A}}{\partial t} + \frac{e}{mc}\mathbf{p}\times\mathbf{H}\right)\frac{\partial N_{\mathbf{qp}}}{\partial\mathbf{p}}} = 0, \tag{13}$$

$$\Delta\overline{\mathbf{A}} - \frac{1}{c^2}\frac{\partial^2\overline{\mathbf{A}}}{\partial t^2} = \frac{1}{c}\frac{\partial}{\partial t}\operatorname{grad}\int\frac{e^2}{|\mathbf{q}-\mathbf{q}'|}\overline{N}_{\mathbf{q}'\mathbf{p}'}\,d\mathbf{q}'\,d\mathbf{p}'$$
$$-\frac{4\pi e}{cm}\int\mathbf{p}'\overline{N}_{\mathbf{qp}'}d\mathbf{p}'. \tag{14}$$

In Eq. (13) the scalar potential φ was eliminated by means of (11).

Averages of the products of random functions are called moments; therefore (13) and (14) are the first and second moments of $N_{\mathbf{qp}}(t)$ and $\mathbf{A}(\mathbf{q}, t)$.

Keeping in mind the relations

$$\overline{N_{\mathbf{qp}}(t)} = Nf_1(\mathbf{q}, \mathbf{p}, t),$$

$$\overline{N_{\mathbf{qp}}(t)\,N_{\mathbf{q}'\mathbf{p}'}(t)} = N(N-1)\,f_2(\mathbf{q}, \mathbf{p}, \mathbf{q}', \mathbf{p}', t)$$
$$+Nf_1(\mathbf{q}, \mathbf{p}, t)\,\delta(\mathbf{q}-\mathbf{q}')\,\delta(\mathbf{p}-\mathbf{p}'),$$

where f_1 and f_2 are the first and second distribution functions, it follows from (13) and (14) that self-consistent equations for the distribution function f_1 and the electric and magnetic field strengths are obtained only when correlation effects can be neglected.

All second moments can be expressed in terms of the following three second moments:

$$M^{(2)} = \overline{N_{\mathbf{qp}}(t)\,N_{\mathbf{q}'\mathbf{p}'}(t')}; \quad S_{\alpha}^{(2)} = \overline{N_{\mathbf{qp}}(t)\,A_{\alpha}(\mathbf{q}', t')};$$
$$A_{\alpha\beta}^{(2)} = \overline{A_{\alpha}(\mathbf{q}, t)\,A_{\beta}(\mathbf{q}', t')}, \quad \alpha, \beta = 1, 2, 3. \tag{15}$$

The equation for $M^{(2)}$ which is obtained by using Eq. (9) for $N_{\mathbf{qp}}(t)$ is the following:

$$\frac{\partial M^{(2)}}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial M^{(2)}}{\partial\mathbf{q}}$$
$$-\frac{\partial}{\partial\mathbf{q}}\int\frac{e^2}{|\mathbf{q}-\mathbf{q}'|}\frac{\partial}{\partial\mathbf{p}}M^{(3)}(\mathbf{q}, \mathbf{p}, t, \mathbf{q}', \mathbf{p}', t', \mathbf{q}'', \mathbf{p}'', t)\,d\mathbf{q}''\,d\mathbf{p}''$$
$$+\overline{\left(\frac{e}{m}\mathbf{E} + \frac{e}{cm}\mathbf{p}\times\mathbf{H}\right)N_{\mathbf{q}'\mathbf{p}'}(t')\frac{\partial N_{\mathbf{qp}}(t)}{\partial\mathbf{p}}} = 0. \tag{16}$$

Equations for $S_{\alpha}^{(2)}$ and $A_{\alpha\beta}^{(2)}$ are easily obtained in a similar manner. Equation (16) and the corresponding equations for $S_{\alpha}^{(2)}$ and $A_{\alpha\beta}^{(2)}$ relate the second and third moments. This chain can be extended. In order to obtain an approximately closed set of equations, say for the second moments, the third moments in (16) and in the corresponding equations for $S_{\alpha}^{(2)}$ and $A_{\alpha\beta}^{(2)}$ must be expressed in terms of second and first moments.

2. In the present paper we shall consider the case of a system of particles with electromagnetic interaction in which a stationary, uniform, random process occurs (see the review articles by Iaglom[8] and Obukhov[9]). Here the second moments $A_{\alpha\beta}^{(2)}$, and the moments $S_{\alpha}^{(2)}$ and $M^{(2)}$ integrated over the momenta, depend only on the absolute values of the time difference $\tau = |t - t'|$ and the coordinate difference $r = |\mathbf{q} - \mathbf{q}'|$. In this case we also have $\overline{N_{\mathbf{qp}}(t)} = \overline{N}_{\mathbf{p}}$, i.e., the average particle distribution is unchanged in time and homogeneous in space, and $\overline{\mathbf{E}} = \overline{\mathbf{H}} = \overline{\mathbf{A}} = 0$. Deviation of the number of particles from the average value is denoted by $\delta N_{\mathbf{qp}}(t)$. Since $\overline{\mathbf{E}} = \overline{\mathbf{H}} = \overline{\mathbf{A}} = 0$ the deviations from the average values of these functions coincide with the functions themselves.

The averages of the products of deviations from average values will, as usual, be called central moments. The subsequent solution will be obtained in the approximation where it is possible to neglect third central moments in an equation for second central moments.

For the purpose of deriving equations for second central moments in the present approximation we first obtain from (9)—(11) the equations for the deviations of $\delta N_{\mathbf{qp}}(t)$ and $\mathbf{A}$ from their mean values:

$$\frac{\partial \delta N_{\mathbf{qp}}}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial \delta N_{\mathbf{qp}}}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}}\int \frac{e^2}{|\mathbf{q}-\mathbf{q}'|}\,\delta N_{\mathbf{q}'\mathbf{p}'}\,d\mathbf{q}'\,d\mathbf{p}'\,\frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}} - \frac{e}{mc}\frac{\partial \mathbf{A}}{\partial t}\frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}} = 0, \tag{17}$$

$$\Delta \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \frac{1}{c}\frac{\partial}{\partial t}\operatorname{grad}\int \frac{e^2}{|\mathbf{q}-\mathbf{q}'|}\,\delta N_{\mathbf{q}'\mathbf{p}'}\,d\mathbf{q}'\,d\mathbf{p}' - \frac{4\pi e}{mc}\int \mathbf{p}'\,\delta N_{\mathbf{qp}'}\,d\mathbf{p}'. \tag{18}$$

In (17) and (18) terms containing products of the deviations $\delta N_{\mathbf{qp}}$ and $\mathbf{A}$ were omitted because these terms in the equations for second central moments give third central moments, which we are neglecting in the present approximation. The term $(e/mc)[\mathbf{p}\times\mathbf{H}]\partial\overline{N}_{\mathbf{p}}/\partial\mathbf{p}$ vanishes because $\overline{N}_{\mathbf{p}} = f(p^2)$.

Three random functions will be of the greatest importance for what follows. These are $n_{\mathbf{q}}(t) = \int N_{\mathbf{qp}}\,d\mathbf{p}$ — the number of particles at a given point of three-dimensional space at a given time, $\mathbf{j}(\mathbf{q},t)$ × $(e/m)\int \mathbf{p}N_{\mathbf{qp}}\,d\mathbf{p}$ — the current, and $\mathbf{A}(\mathbf{q},t)$ — the vector potential. These random functions form one scalar and two vector fields, all of which are stationary, uniform, and isotropic.

In virtue of the condition $\operatorname{div}\mathbf{A} = 0$ the vector field $\mathbf{A}$ is purely rotational. The vector field of the currents can also be divided into potential and rotational parts: $\mathbf{j} = \mathbf{j}^{(p)} + \mathbf{j}^{(r)}$.

According to a theorem proved by Obukhov[9] the correlation functions of the vector field can be represented as the sum of potential and rotational components. Moreover,

$$\overline{n_{\mathbf{q}}(t)\,\mathbf{j}^{(r)}(\mathbf{q}',t')} = 0,$$

i.e., uniform and isotropic scalar and rotational vector fields are not correlated. Finally,

$$\overline{\mathbf{A}\mathbf{j}^{(p)}} = \overline{\mathbf{j}^{(p)}\mathbf{j}^{(r)}} = \overline{\operatorname{grad} n_{\mathbf{q}}\cdot\mathbf{j}^{(r)}} = \overline{\operatorname{grad} n_{\mathbf{q}}\cdot\mathbf{A}} = 0,$$

i.e., homogeneous and isotropic rotational and potential vector fields are not correlated.

It follows that in (18) there can remain only the rotational component of the vector on the right-hand side, i.e.,

$$\Delta \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\mathbf{j}^{(r)}. \tag{19}$$

Equations (17) and (19) will be the basis of our further study.

Before proceding from (17) and (19) to the equations for second moments, we shall divide the former into two parts such that one part defines the potential part of the vector moments and the other defines the rotational part. We assume that the random function $\delta N_{\mathbf{qp}}(t)$ can be divided into two parts: $\delta N_{\mathbf{qp}} = \delta N_{\mathbf{qp}}^{(r)} + \delta N_{\mathbf{qp}}^{(p)}$ which satisfy the equations

$$\frac{\partial \delta N_{\mathbf{qp}}^{(p)}}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial \delta N_{\mathbf{qp}}^{(p)}}{\partial \mathbf{q}} - \frac{\partial}{\partial \mathbf{q}}\int \frac{e^2}{|\mathbf{q}-\mathbf{q}'|}\,\delta N_{\mathbf{q}'\mathbf{p}'}^{(p)}\,d\mathbf{q}'\,d\mathbf{p}'\,\frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}} = 0, \tag{20}$$

$$\frac{\partial \delta N_{\mathbf{qp}}^{(r)}}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial \delta N_{\mathbf{qp}}^{(r)}}{\partial \mathbf{q}} - \frac{e}{mc}\frac{\partial \mathbf{A}}{\partial t}\frac{\partial \overline{N}_{\mathbf{p}}}{\partial \mathbf{p}} = 0,$$

$$\Delta \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c}\int \mathbf{p}\,\delta N_{\mathbf{qp}}^{(r)}\,d\mathbf{p} = -\frac{4\pi}{c}\mathbf{j}^{(r)}. \tag{21}$$

The Hamiltonian function is changed accordingly. Thus in the variables $\mathbf{q}$, $\mathbf{p}$, $\mathbf{A}$, $\dot{\mathbf{A}}$ the expression for the departure of the Hamiltonian function from its average value is

$$\delta\mathscr{H} = \int \frac{p^2}{2m}\left(\delta N_{\mathbf{qp}}^{(p)} + \delta N_{\mathbf{qp}}^{(r)}\right) d\mathbf{q}\,d\mathbf{p}$$

$$+ \frac{1}{2}\int \frac{e^2}{|\mathbf{q}-\mathbf{q}'|}\,\delta N_{\mathbf{qp}}^{(p)}\,\delta N_{\mathbf{q}'\mathbf{p}'}^{(p)}\,d\mathbf{q}'\,d\mathbf{p}'\,d\mathbf{q}\,d\mathbf{p}$$

$$+ \frac{1}{8\pi}\int\left\{\left(\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right)^2 + (\operatorname{curl}\mathbf{A})^2\right\} d\mathbf{q}. \tag{22}$$

The foregoing partition of the initial equations is somewhat analogous to the partition of the set of self-consistent equations for the distribution function f_1 and the electric and magnetic fields into rotational and longitudinal parts, as was done by Vlasov in Ref. 7 and elsewhere.

In the present case this partition is justified by the fact that the second moments calculated by means of the random function $\delta N_{\mathbf{qp}}^{(p)} + \delta N_{\mathbf{qp}}^{(r)}$ separate into the sum of uncorrelated potential and rotational components.

We have, for example,

$$\overline{\delta n_{\mathbf{q}}(t)\,\delta n_{\mathbf{q}'}(t')} = \overline{\delta n_{\mathbf{q}}^{(p)}(t)\,\delta n_{\mathbf{q}'}^{(p)}(t')}$$

$$+ \overline{\delta n_{\mathbf{q}}^{(p)}(t)\,\delta n_{\mathbf{q}'}^{(p)}(t')} + \overline{2\delta n_{\mathbf{q}}^{(p)}(t)\,\delta n_{\mathbf{q}'}^{(r)}(t')}.$$

Let us consider the last term. In virtue of the stationary, uniform, and isotropic conditions we have

$$\overline{\delta n_{\mathbf{q}}^{(p)}(t)\,\delta n_{\mathbf{q}'}^{(r)}(t')} = \Pi(\tau, r).$$

Using the continuity equations

$$\frac{\partial}{\partial t}\delta n_{\mathbf{q}}^{(p)} + \operatorname{div}\mathbf{j}^{(p)} = 0, \quad \operatorname{div}\mathbf{j}^{(r)} = 0, \quad \frac{\partial}{\partial t}\delta n_{\mathbf{q}}^{(r)} = 0,$$

we obtain $\partial\Pi(\tau, r)/\partial\tau = 0$ and thus $\Pi(\tau, r) =$

$\Pi(0, \mathbf{r})$. But from (22) it follows that $\Pi(0, \mathbf{r}) = 0$, so that the term in question really vanishes.

The separation of the correlation function for currents into rotational and potential components follows directly from Obukhov's theorem. The same holds true for $\overline{\mathbf{j}(\mathbf{q}, t)\,\mathbf{A}(\mathbf{q}', t')}$.

Let us now consider the correlation of the scalar and vector fields:

$$\overline{\delta n_{\mathbf{q}}(t)\,\mathbf{j}(\mathbf{q}, t')} = \overline{\delta n_{\mathbf{q}}^{(p)}\,\mathbf{j}^{(p)}} + \overline{\delta n_{\mathbf{q}}^{(r)}\,\mathbf{j}^{(r)}} + \overline{\delta n_{\mathbf{q}}^{(p)}\,\mathbf{j}^{(r)}} + \overline{\delta n_{\mathbf{q}}^{(r)}\,\mathbf{j}^{(p)}}.$$

The third term vanishes because the uniform and isotropic scalar and rotational vector fields are not correlated. The last term in this case can be represented by

$$\overline{\delta n_{\mathbf{q}}^{(r)}\,\mathbf{j}_{\alpha}^{(p)}(\mathbf{q}', t')} = B(\tau, r)\,x_{\alpha}/r.$$

Using the continuity equations, we obtain

$$\operatorname{div}_{\mathbf{q}'}\,\delta n_{\mathbf{q}}^{(r)}(t)\,j_{\mathbf{q}'}^{(p)}(t') = 0.$$

Consequently,

$$\frac{\partial}{\partial x_{\alpha}}(Bx_{\alpha}/r) = \frac{2B}{r} + B' = 0,$$

whence it follows that $B = C/r^2$, where C is a constant of integration. From the boundary condition of the correlation function at zero $C = 0$ and $B = 0$.

3. We now obtain the equations for second moments and investigate the solutions of these equations.

Equation (20) agrees with the corresponding equation in Ref. 6, where equations for the correlation functions of a centrally-interacting particle system were considered. Multiplying (20) by $\delta N_{\mathbf{q}'\mathbf{p}'}(t')$ and averaging, we obtain an equation for $M(\mathbf{r}, \tau, \mathbf{p}, \mathbf{p}')$. The superscript "p" can be omitted here.

$$\frac{\partial M}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial M}{\partial \mathbf{q}}$$

$$-\frac{\partial}{\partial \mathbf{q}}\int \frac{e^2}{|\mathbf{q} - \mathbf{q}'|} M(\mathbf{q}' - \mathbf{q}'', \tau, \mathbf{p}'', \mathbf{p}')\,d\mathbf{q}''\,d\mathbf{p}'\,\frac{\partial \bar{N}_{\mathbf{p}}}{\partial \mathbf{p}} = 0. \quad (23)$$

Expanding M in a Fourier integral with respect to $\mathbf{r}$, and using a Laplace transform with respect to τ, we have

$$M = \frac{1}{(2\pi)^4 i}\iint M_{\mathbf{k}s}(p, p')\,e^{s\tau - i\mathbf{kr}}\,ds\,dk \quad (24)$$

after which, when the solution is found by Landau's method in Ref. 10, we arrive at the following expression:

$$\int M(\mathbf{r}, \tau, \mathbf{q}, \mathbf{p}')\,d\mathbf{p}\,d\mathbf{p}' \equiv M(\tau, \mathbf{r})$$

$$= \frac{1}{(2\pi)^4 i}\iint \frac{\displaystyle\int \frac{M_{\mathbf{k}}(0, \mathbf{p})}{s - i\mathbf{kp}/m}\,d\mathbf{p}}{1 + \dfrac{4\pi e^2}{k^2}\displaystyle\int \frac{i\mathbf{k}\,\partial N_{\mathbf{p}}/\partial \mathbf{p}}{s - i\mathbf{kp}/m}\,d\mathbf{p}}\,e^{s\tau - i\mathbf{kr}}\,ds\,dk. \quad (25)$$

Equation (25) agrees with Tolmachev's solution.[4] The solution of (25) expresses $M(\tau, \mathbf{r})$ in terms of the correlation function for $\tau = 0$:

$$g(r) = M(0, r)/\bar{n}^2 - \delta(r)/\bar{n}; \quad \bar{n} = \int \bar{N}_{\mathbf{p}}\,d\mathbf{p}.$$

An expression for this function is known in some cases.

We now consider Eqs. (21) for $\delta N_{\mathbf{qp}}^{(r)}(t)$ and $\mathbf{A}$. Multiplying both equations by $\delta N_{\mathbf{q}'\mathbf{p}'}^{(r)}(t)$ and averaging, we obtain for the functions

$$M^{(r)} = \overline{\delta N_{\mathbf{qp}}^{(r)}(t)\,\delta N_{\mathbf{q}'\mathbf{p}'}^{(r)}(t')}, \quad \mathbf{S}'^{(r)} = \overline{\delta N_{\mathbf{q}'\mathbf{p}'}(t')\,\mathbf{A}(\mathbf{q}, t)}$$

the closed set of equations

$$\frac{\partial M^{(r)}}{\partial t} + \frac{\mathbf{p}}{m}\frac{\partial M^{(r)}}{\partial \mathbf{q}} - \frac{e}{mc}\frac{\partial \mathbf{S}^{(r)}}{\partial t}\frac{\partial \bar{N}_{\mathbf{p}}}{\partial \mathbf{p}} = 0, \quad (26)$$

$$\Delta \mathbf{S}^{(r)} - \frac{1}{c^2}\frac{\partial^2 \mathbf{S}^{(r)}}{\partial t^2} = -\frac{4\pi}{c}\frac{e}{m}\int \mathbf{p}' M^{(r)}\,d\mathbf{p}'. \quad (27)$$

When in these equations we represent $M^{(r)}$ and $\mathbf{S}^{(r)}$ by Fourier integrals with respect to the coordinates and perform a Laplace transformation with respect to time [see Eq. (24)], we obtain the following equations for the transformation amplitudes:

$$\left(s - i\frac{\mathbf{kp}}{m}\right) M_{\mathbf{k}s}^{(r)}(\mathbf{p}, \mathbf{p}') - \frac{es}{mc}\,\mathbf{S}_{\mathbf{k}s}^{(r)}(\mathbf{p}')\frac{\partial \bar{N}_{\mathbf{p}}}{\partial \mathbf{p}}$$

$$= M_{\mathbf{k}}^{(r)}(0, \mathbf{p}, \mathbf{p}') - \frac{e}{mc}\,\mathbf{S}_{\mathbf{k}}^{(r)}(0, \mathbf{p}')\frac{\partial \bar{N}_{\mathbf{p}}}{\partial \mathbf{p}}, \quad (28)$$

$$\left(\frac{s^2}{c^2} + k^2\right)\mathbf{S}_{\mathbf{k}s}^{(r)}(\mathbf{p}') = \frac{4\pi e}{cm}\int \mathbf{p}'' M_{\mathbf{k}s}^{(r)}(\mathbf{p}', \mathbf{p}'')\,d\mathbf{p}''$$

$$+ \frac{s}{c^2}\mathbf{S}_{\mathbf{k}}^{(r)}(0, \mathbf{p}') + \frac{1}{c^2}\left(\frac{\partial}{\partial t}\mathbf{S}_{\mathbf{k}}^{(r)}(\tau, \mathbf{p}')\right)_{\tau=0}. \quad (29)$$

Eliminating $\mathbf{S}_{\mathbf{k}s}^{(r)}(\mathbf{p}')$ in these equations, we obtain

$$\left(s - \frac{i\mathbf{kp}}{m}\right) M_{\mathbf{k}s}^{(r)}(\mathbf{p}, \mathbf{p}') - \frac{4\pi e^2 s}{m(s^2 + c^2k^2)}\int \mathbf{p}'' M_{\mathbf{k}s}^{(r)}(\mathbf{p}', \mathbf{p}'')\,d\mathbf{p}''\,\frac{\partial \bar{N}_{\mathbf{p}}}{\partial \mathbf{p}}$$

$$= M_{\mathbf{k}}(0, \mathbf{p}, \mathbf{p}') \quad (30)$$

$$+ \frac{e}{mc(s^2 + c^2k^2)}\left\{s\frac{\partial}{\partial t}\mathbf{S}^{(r)}(\tau, \mathbf{p}') - c^2k^2\mathbf{S}^{(r)}(\tau, \mathbf{p}')\right\}_{\tau=0}\frac{\partial \bar{N}_{\mathbf{p}}}{\partial \mathbf{p}}.$$

We now divide (30) by $s - i\mathbf{kp}/m$, multiply by $\mathbf{pp}'$ and integrate over $\mathbf{p}, \mathbf{p}'$. We direct the vector $\int \mathbf{p}' M_{\mathbf{k}s}^{(r)}(\mathbf{p}'', \mathbf{p}')\,d\mathbf{p}'$ along the y axis. For the divergence of the vector we have

$$\operatorname{div}_{\mathbf{q}'}\int \mathbf{p}' M^{(r)}(\mathbf{q}'', \mathbf{q}', \mathbf{p}'', \mathbf{p}')\,d\mathbf{p}' = 0,$$

since $\operatorname{div}_{\mathbf{q}}\,\mathbf{j}^{(r)}(\mathbf{q}, t) = 0$. Therefore $\mathbf{k}\int \mathbf{p}' M_{\mathbf{k}s}^{(r)}(\mathbf{p}'', \mathbf{p}')\,d\mathbf{p}' = 0$ and, consequently, the vector $\mathbf{k}$ can be directed along the x axis. Then the integral of the second term in the left-hand side of (30) can be represented as

$$\frac{4\pi e^2 s}{m^2(s^2+c^2k^2)\varkappa T}\int \frac{p_y\, \mathrm{p}'\, (\mathrm{p}''\mathrm{p})\, M_{\mathrm{k}s}^{(\mathrm{r})}(\mathrm{p}',\mathrm{p}'')}{s - ikp_x / m}\, \overline{N}_{\mathrm{p}}\, d\mathrm{p}\, d\mathrm{p}'\, d\mathrm{p}''. \quad (31)$$

In (31) the integrals which contain the products $p_x p_x''$ and $p_z p_z''$ vanish since $\overline{N}_p = f(p^2)$. Using the notation

$$\int \mathrm{pp}'\, M_{\mathrm{k}s}^{(\mathrm{r})}(\mathrm{p},\mathrm{p}')\, d\mathrm{p}\, d\mathrm{p}' = \overline{(\mathrm{j}(\mathrm{q},t)\,\mathrm{j}(\mathrm{q}',t'))}_{\mathrm{k}s},$$

we obtain

$$(\mathrm{j}^{(\mathrm{r})}\mathrm{j}^{(\mathrm{r})})_{\mathrm{k}s} = \frac{\displaystyle\int \frac{\mathrm{pp}'\, M_{\mathrm{k}}^{(\mathrm{r})}(0;\mathrm{p},\mathrm{p}')}{s - ik\mathrm{p}/m}\, d\mathrm{p}\, d\mathrm{p}' + \frac{e/mc}{s^2+k^2c^2}\int(\mathrm{pp}')\, \frac{\partial \overline{N}_{\mathrm{p}}}{\partial \mathrm{p}}\left\{s\frac{\partial}{\partial t}\, \mathrm{S}_{\mathrm{k}}^{(\mathrm{r})}(\tau,\mathrm{p}') - c^2k^2 \mathrm{S}_{\mathrm{k}}^{(\mathrm{r})}(\tau,\mathrm{p}')\right\}_{\tau=0} d\mathrm{p}\, d\mathrm{p}'}{\displaystyle 1 + \frac{4\pi e^2 s}{m^2(s^2+c^2k^2)\varkappa T}\int \frac{p_y^2\, \overline{N}_{\mathrm{p}}}{s - ikp_x/m}\, d\mathrm{p}}. \quad (32)$$

From (32) we obtain an expression for the current correlation function:

$$\overline{\mathrm{j}^{(\mathrm{r})}(\mathrm{q},t)\, \mathrm{j}^{(\mathrm{r})}(\mathrm{q}',t')} = \frac{1}{(2\pi)^4 i}\iint (\mathrm{j}^{(\mathrm{r})}\mathrm{j}^{(\mathrm{r})})_{\mathrm{k}s}\, e^{s\tau - i\mathrm{kr}}\, ds\, d\mathrm{k}. \quad (33)$$

This relation expresses the space-time correlation function for currents in terms of simultaneous correlation functions.

From a comparison of (25) and (33) it follows that the space-time density correlation function is largely determined by the form of the expression

$$1 + \frac{4\pi e^2}{k^2}\int \frac{i\mathrm{k}\, \partial \overline{N}_{\mathrm{p}}/\partial \mathrm{p}}{s - i\mathrm{kp}/m}\, d\mathrm{p}, \quad (34)$$

which is the dielectric constant of a medium in which random longitudinal oscillations are occurring. When (34) is equated to zero we obtain the dispersion equation which was investigated by Vlasov[7] and Landau.[10]

From (33) it follows that the space-time correlation function for eddy currents is determined by

$$1 + \frac{4\pi e^2 s}{(s^2+c^2k^2)\, m^2 \varkappa T}\int \frac{p_y^2\, N_{\mathrm{p}}}{s - ikp_x/m}\, d\mathrm{p}. \quad (35)$$

It will follow subsequently that (35) is associated with the dielectric constant of a medium in which random transverse oscillations are taking place.* When this expression is equated to zero we obtain the dispersion equation for transverse oscillations which was considered by Vlasov.[7]

By setting the real part of (34) equal to zero we obtain the frequency of longitudinal random oscillations. For example, for long waves the frequency and decrement σ_k of random longitudinal oscillations are given by the formulas[10]

$$\omega^2 = \omega_L^2 + 3r_d^2 k^2, \quad \sigma_k = \omega_L\sqrt{\frac{\pi}{8}}\,(kr_d)^{-3}\, e^{-k^2 r^2{}_d/2}$$

where r_d is the Debye radius.

The corresponding formulas for (35) when $\omega_k/k \gg \sqrt{\kappa T/m}$ are

$$\omega_k^2 = \omega_L^2 + c^2k^2; \quad \sigma_k = \frac{\pi\omega_L^2}{2kn}(\overline{N}_{p_x})_{p_x = \omega_k/k}. \quad (36)$$

It will become clear from what follows that the magnitude of σ_k is associated with the imaginary part of the dielectric constant for transverse oscillations. Since $N_{p_x} = 0$ for $p_x > mc$ it follows from (36) that $\sigma_k = 0$ when $\omega_k/k > c$, i.e., the damping vanishes for waves whose phase velocity is greater than the speed of light. These waves cannot be excited by thermal motion and thus do no contribute to thermal fluctuations. This fact limits the wavelengths of random transverse oscillations. For example, oscillations with the frequency $\omega_k \approx \omega_L$ which contribute to thermal oscillations can have only wave numbers for which $k \geq \omega_L/c = 1/\delta$, where $\delta = (mc^2/4\pi e^2\overline{n})^{1/2}$. These results can be obtained in a more consistent manner by using the relativistic equation for $N_{qp}(t)$.

4. In this section we shall derive expressions for the space-time correlation function of the vector potentials $A(r,\tau) = \overline{A(q',t')\, A(q,t)}$, by means of which we can obtain correlation functions for the electric and magnetic field strengths. We shall then establish the correspondence of these results with the expressions for the correlation functions of extraneous currents and electric and magnetic fields that were obtained phenomenologically by Leontovich and Rytov.[12,1] Multiplying (21) by $A(q',t')$, averaging, expanding into a Fourier integral with respect to r, and taking the Laplace transform with respect to τ, we obtain for the functions

$$\mathrm{S}_{\mathrm{k}s}(\mathrm{p}) = \int_0^\infty\int \overline{\mathbf{A}(\mathrm{q}',t')\, \delta N_{\mathrm{qp}}^{(\mathrm{r})}(t)}\, e^{-s\tau + i\mathrm{kr}}\, d\tau\, d\mathbf{r}, \quad A_{\mathrm{k}s},$$

the following equations

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*Expressions for the dielectric constant of a plasma, based on equations involving a self-consistent field, were studied by Gertsenshtein.[11]

$$\left(s - \frac{i\mathbf{kp}}{m}\right)\mathbf{S}_{\mathbf{k}s}(\mathbf{p}) - \frac{es}{mc} A_{\mathbf{k}s} \frac{\partial \bar{N}_{\mathbf{p}}}{\partial \mathbf{p}} = \mathbf{S}_{\mathbf{k}}(\tau = 0, \mathbf{p}) - \frac{e}{mc} \overline{\{\mathbf{A}(\mathbf{q}', t')\,\dot{\mathbf{A}}_i(\mathbf{q}, t)\}}_{\mathbf{k},\, \tau=0} \frac{\partial \bar{N}_{\mathbf{p}}}{\partial \mathbf{p}}, \qquad (37)$$

$$\left(k^2 + \frac{s^2}{c^2}\right) A_{\mathbf{k}s} = \frac{4\pi e}{mc} \int \mathbf{p}\mathbf{S}_{\mathbf{k}s}(\mathbf{p})\, d\mathbf{p} + \frac{s}{c^2} A_{\mathbf{k}}(\tau = 0) + \frac{1}{c^2} \overline{(\mathbf{A}(\mathbf{q}', t')\,\dot{\mathbf{A}}(\mathbf{q}, t))}_{\mathbf{k},\, \tau=0}. \qquad (38)$$

If $\mathbf{S}_{\mathbf{k}s}(\mathbf{p})$ is eliminated from these equations and we perform transformations similar to those employed in deriving (32), we obtain the following equation for $A_{\mathbf{k}s}$:

$$\left\{k^2 + \frac{s^2}{c^2}\left(1 + \frac{4\pi e^2}{sm^2\varkappa T} \int \frac{N_{\mathbf{p}} p_y^2}{s - ikp_x/m}\, d\mathbf{p}\right\} A_{\mathbf{k}s} = \frac{1}{c^2} \overline{(\mathbf{A}(\mathbf{q}', t')\,\dot{\mathbf{A}}(\mathbf{q}, t))}_{\mathbf{k},\, \tau \cdot 0} + \frac{s}{c^2}\left(1 + \frac{4\pi e^2}{sm^2\varkappa T} \int \frac{N_{\mathbf{p}} p_y^2}{s - ikp_x/m}\, d\mathbf{p}\right) A_{\mathbf{k}}(\tau = 0) + \frac{4\pi e}{mc} \int \frac{\mathbf{p}\mathbf{S}_{\mathbf{k}s}}{s - i\mathbf{kp}/m}\, d\mathbf{p}. \qquad (39)$$

When the left-hand side of (39) is equated to the corresponding part of the equation for a medium of dielectric constant ϵ we obtain

$$\varepsilon_{\mathbf{k}s} = 1 + \frac{4\pi e^2}{sm^2\varkappa T} \int \frac{\bar{N}_{\mathbf{p}}}{s - ikp_x/m}\, d\mathbf{p}. \qquad (40)$$

Equation (39) can then be written as

$$\left(k^2 + \frac{s^2\varepsilon_{\mathbf{k}s}}{c^2}\right) A_{\mathbf{k}s} = \frac{s\varepsilon_{\mathbf{k}s}}{c^2} A_{\mathbf{k}}(\tau = 0) + \frac{4\pi e}{mc} \int \frac{\mathbf{p}S_{\mathbf{k}s}(\mathbf{p})}{s - i\mathbf{kp}/m}\, d\mathbf{p} + \frac{1}{c^2} \overline{(\mathbf{A}(\mathbf{q}', t')\,\dot{\mathbf{A}}(\mathbf{q}, t))}_{\mathbf{k},\, \tau=0}. \qquad (41)$$

This enables us to obtain

$$A(\tau, r) = \overline{\mathbf{A}(\mathbf{q}', t')\,\mathbf{A}(\mathbf{q}, t)} = \frac{1}{(2\pi)^4 i} \iint A_{\mathbf{k}s} e^{s\tau - i\mathbf{kr}}\, ds\, d\mathbf{k}. \qquad (42)$$

Equations (41) and (42) express in general form the space-time function of the vector potentials by means of equilibrium correlation functions (correlation functions for $\tau = 0$).

We now turn to the comparison of (41) and (42) with the corresponding expressions which were obtained phenomenologically by Rytov (see also the book of Landau and Lifshitz[13]), and ascertain under what conditions is it possible to derive the expressions given by the latter for the correlation functions of extraneous random electric and magnetic fields.

We first take it into account that the dielectric constant is a complex quantity: $\epsilon_{\mathbf{k}s} = \epsilon'_{\mathbf{k}s} + i\epsilon''_{\mathbf{k}s}$. It follows from (40) that

$$\varepsilon'_{k\omega} = 1 - \frac{\omega_L^2}{\omega_L^2 + c^2k^2}, \quad \text{for } \frac{\omega_k}{k} \gg \sqrt{\frac{\varkappa T}{m}}, \ \varepsilon''_{k\omega} = 2\sigma_k/\omega. \qquad (43)$$

$\epsilon'_{\mathbf{k}\omega}$ and $\epsilon''_{\mathbf{k}\omega}$ depend generally on both ω and $\mathbf{k}$, i.e., both temporal and spatial dispersion occur. The space-time correlation function diminishes quite rapidly with increasing τ and $\mathbf{r}$, so that instead of a Laplace transformation it is possible to use the one-sided Fourier transformation. The coefficients of the expansion determine the energy distribution of the thermal oscillations with respect to frequencies and wave numbers. In (41) we now replace s by $i\omega$ [as was done in deriving (43)] and assume $\mathbf{A}\dot{\mathbf{A}} = 0$, $\mathbf{pS}(\tau = 0, \mathbf{r}, \mathbf{p}) = 0$. Then (41) can be written as

$$(c^2k^2 - \omega_k^2\varepsilon_{\mathbf{k}\omega}) A_{\mathbf{k}\omega} = i\omega\varepsilon_{\mathbf{k}\omega} A_{\mathbf{k}}(\tau = 0). \qquad (44)$$

From a comparison of (44) with the corresponding equation in Rytov's paper[1] we conclude that the right-hand side of this equation represents the coefficients in the Fourier integral for the space-time correlation function of the extraneous electric induction $\mathbf{D}$. Separating the real part, we obtain

$$\overline{\mathbf{D}_{\mathbf{k}\omega}\mathbf{D}^*_{\mathbf{k}\omega}} = \frac{2\omega\varepsilon''_{\mathbf{k}\omega}}{c^2} A_{\mathbf{k}}(\tau = 0). \qquad (45)$$

When the spatial dispersion can be neglected, (45) becomes

$$\overline{\mathbf{D}_{\mathbf{k}\omega}\mathbf{D}^*_{\mathbf{k}\omega}} = 8\pi \frac{\varepsilon''_{\omega}}{\omega} \varkappa T. \qquad (45')$$

We have considered that in this case $A_{\mathbf{k}}(\tau = 0) = 4\pi\kappa T/\omega^2$. From (45′) we obtain

$$\overline{\mathbf{D}_{\omega}(\mathbf{q})\,\mathbf{D}_{\omega}(\mathbf{q}')}/8\pi = \frac{\varepsilon''_{\omega}}{\omega} \varkappa T \delta(\mathbf{q} - \mathbf{q}'). \qquad (46)$$

Equations (46) and (45) agree with the corresponding equations of Rytov.

From the comparison of these results we see that the microscopic approach permits us to obtain an explicit expression for the dielectric constant and to derive more general relations which are also valid when spatial dispersion is present.

By a similar comparison in a quasi-stationary approximation we obtain an expression for the space-time spectral function of extraneous currents:

$$\overline{(\mathbf{j}_{\mathbf{k}\omega}\mathbf{j}^*_{\mathbf{k}\omega})}_{\mathrm{extr}} = 2\omega k^2 \varepsilon''_{\mathbf{k}\omega} A_{\mathbf{k}}(\tau = 0)/(4\pi)^2. \qquad (47)$$

When spatial dispersion can be neglected we obtain from (47)

$$\overline{(\mathbf{j}_{\omega\mathbf{k}}\mathbf{j}_{\omega\mathbf{k}})}_{\mathrm{extr}} = 2\omega\varepsilon''_{\omega}\varkappa T/4\pi, \qquad (48)$$

$$(\mathbf{j}_{\omega}(q)\,\mathbf{j}^*_{\omega}(\mathbf{q}'))_{\mathrm{extr}} = 2\sigma\delta(\mathbf{q} - \mathbf{q}').$$

Here $\epsilon'' = 4\pi\sigma/\omega$ and σ is the conductivity. Equation (48) agrees with the expression obtained by Leontovich and Rytov.[12]

We note once more that by the present method there is no need to introduce the concepts of "extraneous" random currents and fields, since all space-time correlation functions can be expressed in terms of simultaneous correlation functions. The correlation functions of extraneous currents and fields in the phenomenological theory are equivalent to approximate simultaneous correlation functions.

Using (9) — (11) for the random functions $\delta N_{\mathbf{qp}}(t)$, $\mathbf{A}$, and φ, we can also obtain equations for the equilibrium correlation functions. In the approximation where third central moments are dropped, the equations for simultaneous correlation functions can be obtained from (20) and (21). Thus, for example, the equation for the correlation function $M^{(\mathbf{p})}(\mathbf{r}, 0, \mathbf{p}, \mathbf{p}')$ in the present approximation is obtained from (20) in the form

$$\frac{\mathbf{p}}{m}\frac{\partial M^{(\mathbf{p})}}{\partial \mathbf{q}} + \frac{\mathbf{p}'}{m}\frac{\partial M^{(\mathbf{p})}}{\partial \mathbf{q}'}$$
$$- \frac{\partial}{\partial \mathbf{q}} \int \frac{e^2}{|\mathbf{q} - \mathbf{q}''|} M^{(\mathbf{p})}(\mathbf{q}'' - \mathbf{q}', \mathbf{p}', \mathbf{p}'')\, d\mathbf{q}''\, d\mathbf{p}'' \frac{\partial \bar{N}_{\mathbf{p}}}{\partial \mathbf{p}} \qquad (49)$$
$$- \frac{\partial}{\partial \mathbf{q}'} \int \frac{e^2}{|\mathbf{q}' - \mathbf{q}''|} M^{(\mathbf{p})}(\mathbf{q}'' - \mathbf{q}, \mathbf{p}, \mathbf{p}'')\, d\mathbf{q}''\, d\mathbf{p}'' \frac{\partial \bar{N}_{\mathbf{p}'}}{\partial \mathbf{p}'} = 0.$$

This equation agrees with the corresponding equation obtained in Bogoliubov's book[14] by expansion with respect to a plasma parameter. The correlation function which is obtained from the solution of (49) agrees with the Debye correlation function.

By means of (21) or the corresponding equations in the variables $\mathbf{q}$, $\mathbf{P}$, we can obtain equations for the equilibrium correlation functions of the rotational fields which have been considered. This is an independent problem and will be considered separately.

In conclusion we shall make two additional comments. Vector fields are characterized by a correlation tensor. For homogeneous and isotropic fields the correlation tensor is entirely determined by two scalar functions. We have obtained above expressions for the diagonal element sums of correlation tensors for the current and the vector potential. The diagonal element sum of a correlation tensor is expressed in terms of the two scalar functions which determine the tensor. A second equation that relates these functions is the Kármán condition for random rotational fields. These formulas thus enable us to obtain the correlation tensor.

The solution that has been given above is valid whenever third central moments can be neglected. This is a poor approximation when collisions between particles of the system play an important part. However, the applicability of the foregoing formulas can be extended considerably if collisions are taken into account by introducing into the right-hand side of (20) and of the first equation in (21) the terms

$$-\nu\delta N^{(\mathbf{p})}_{\mathbf{qp}}(t), \qquad -\nu\delta N^{(\mathbf{r})}_{\mathbf{qp}}(t),$$

in which ν is the frequency of the collisions. In this approximation all of the formulas derived above remain valid when $s - i\mathbf{kp}/m$ is replaced by $s + \nu - i\mathbf{kp}/m$. For example, the expression for the dielectric constant becomes

$$\varepsilon_{\mathbf{ks}} = 1 + \frac{4\pi e^2}{sm^2\varkappa T} \int \frac{\bar{N}_{\mathbf{p}} p_y^2}{s + \nu - ikp_x/m}\, d\mathbf{p}.$$

The method which has been described can be applied to the study of a quantized system of particles with electromagnetic interaction and for a system of electrons interacting with lattice vibrations. It is thus possible, in particular, to obtain corresponding microscopic expressions for the relations of Callen and Welton.

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STABILITY OF SHOCK WAVES IN RELATIVISTIC HYDRODYNAMICS

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Stability against small perturbations of the discontinuity surface is investigated for shock waves in an arbitrary medium, described by relativistic equations for an ideal fluid.[*]

1. INTRODUCTION

THE concept of an ideal fluid is applicable in two limiting cases of relativistic hydrodynamics: at sufficiently low temperatures, when the mean number of produced pairs is much smaller than the number of virtual particles, and in the ultra-relativistic case of super-high temperatures, when the mean number of pairs is much larger than the number of virtual particles. In fact, it follows from the equations[†]

$$\partial T_i^k / \partial x^k = 0, \qquad T_{ik} = w u_i u_k + p g_{ik} \tag{1.1}$$

that the entropy flux density satisfies the equation

$$\frac{\partial \sigma^i}{\partial x^i} = -\frac{\mu}{T} \frac{\partial n^i}{\partial x^i}, \quad \sigma^i = \sigma u^i, \; n^i = n u^i. \tag{1.2}$$

In the first case, which we shall call relativistic, the equation of continuity holds for the number of particles in zero approximation of the ratio of the mean number of pairs to the number of particles

$$\partial n^i / \partial x^i = 0. \tag{1.3}$$

In the ultra-relativistic limit the chemical potential is equal to zero in zeroth approximation of the ratio of virtual particles to the number of pairs:

$$\mu = 0. \tag{1.4}$$

In both limiting cases (and only then), the entropy is conserved:

$$\partial \sigma^i / \partial x^i = 0. \tag{1.5}$$

It should be noted that, as shown by Khalatnikov,[3] it is possible to obtain the equations for the ultra-relativistic case, (1.1) and (1.5) from Eqs. (1.1) and (1.3) of the relativistic case by a simple substitution:

$$w \to T\sigma, \; n \to \sigma, \tag{1.6}$$

and putting $\mu = 0$.[*] We shall make use of this result later.

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[*] In classical hydrodynamics this problem was solved by D'iakov.[1]

[†] Our notation follows Ch. XV of the book by Landau and Lifshitz.[2]

[*] Thermodynamical relations necessary for the completeness of the system (with exception of the equation of state) remain valid after the substitution (1.6), in view of Eq. (1.4): if $\mu = 0$, n does not enter into the thermodynamical identities, and $w = T\sigma$. The equations for the ultra-relativistic case can therefore be obtained at any stage from the relativistic equations if one does not use the equation of state explicitly. If the boundary conditions are obtained directly from the equations, or if there is no condition imposed on n at the boundary, then the above procedure permits us to obtain the corresponding solution for the ultra-relativistic case from the solution of the boundary problem.

If we note that conditions at hydrodynamic discontinuities do not follow from equations of the ideal fluid, but represent additional physical requirements (following from the equations with dissipation), it becomes clear that the substitution (1.6) is applicable to tangential and is inapplicable to normal discontinuities, since n enters the boundary conditions for the latter.

It should also be noted that considerations of stability of shock waves against motion of the surfact of discontinuity as a whole, which lead to the following inequalities in the non-relativistic case:

$$v < s, \quad \bar{v} > \bar{s}, \tag{1.7}$$

are fully valid in relativistic hydrodynamics. Here v and $\bar{v}$ denote the component normal to the surface of discontinuity of the three-dimensional hydrodynamical velocity in the system where the discontinuity is at rest, while s and $\bar{s}$ is the velocity of sound in the eigen-system; the bar denotes values in the part of the medium before the oncoming shock wave.

In fact, according to Einstein's formula, we have for the velocity of the disturbance in the system where the discontinuity is at rest,

$$(v + s)/(1 + vs), \quad (\bar{v} + \bar{s})/(1 + \bar{v}\bar{s}).$$

The sign of the above expressions is determined by the numerator only, i.e., the choice of the sign determining the possibility or impossibility of propagation of the disturbance from the surface of discontinuity leads to the same inequalities as obtained from the Galileo transformation. It is necessary for the sake of stability that the number of parameters describing the perturbation should not exceed the number of equations. The inequalities (1.7) lead to such a minimum number of parameters both in the non-relativistic[2] and relativistic and ultra-relativistic cases. It should be noted that, although the number of equations in the ultra-relativistic case is smaller by one, the number of parameters decreases also by the same amount since there are no separate sound and entropy disturbances.

2. VARIATIONAL EQUATIONS OF MOTION

According to (1.7), the motion of a shock wave should be supersonic in the part of the medium into which it enters, and subsonic in the region left behind. This means that disturbances of the surface of discontinuity will influence only the flow behind the wave. Let us choose the axis $y = x_2$ in the direction of the normal to the surface of discontinuity, aligned with the hydrodynamical velocity v, and the axis $x = x_1$ in the direction of the wave vector k_1 of the disturbance, which will be specified as a traveling wave of small amplitude η:

$$y = \eta \exp i\,(k_1 x^1 + k_0 x^0). \tag{2.1}$$

Variation δA of an arbitrary variable A behind the shock wave will be of the form of a traveling wave with wave vectors k_1 and k_2 and frequency k_0

$$\delta A \sim \exp(ik_l x^l), \quad \partial\delta A/\partial x_l = ik_l \delta A.$$

Next we shall write linearized equations for the amplitudes:

$$k_l \delta T_i^l = 0; \quad \delta T_i^l = \delta w u_i u^l + w \delta u_i u^l + w u_i \delta u^l; \tag{2.2}$$

$$k_l \delta n^l = 0; \; \delta n^l = \delta n u^l + n \delta u^l. \tag{2.3}$$

Multiplying Eq. (2.2) by u^i, and making use of Eq. (2.3) and of the thermodynamical identity

$$d\frac{w}{n} = Td\frac{\sigma}{n} + \frac{1}{n}dp, \tag{2.4}$$

we obtain easily

$$k_l u^l \delta \frac{\sigma}{n} = 0. \tag{2.5}$$

It is convenient to transform Eq. (2.2) into

$$k_l u^l \{u_i \delta p + w \delta u_i\} + k_i \delta p = 0\,. \tag{2.6}$$

It can be seen from Eq. (2.5) that there are two types of solutions. The first represents turbulent entropy waves carried by the liquid current

$$k_l^{(1)} u^l = 0. \tag{2.7}$$

It follows immediately from (2.3) and (2.6) that

$$k_l^{(1)} \delta u^{(1)l} = 0, \quad \delta p^{(1)} = 0. \tag{2.8}$$

The second type of solutions (disturbances of the sound-wave type) is

$$k_l^{(2)} u^l \neq 0. \tag{2.9}$$

Omitting the superscripts in $k_1^{(2)}$ and $\delta p^{(2)}$, we have

$$\delta u_i^{(2)} = -\delta p \frac{u_i k_l u^l + k_i}{w k_l u^l}\,. \tag{2.10}$$

Multiplying Eq. (2.6) by k_i we obtain an equation for $k^{(2)}$

$$(k_l u^l)^2 (1 - s^{-2}) + k_l k^l = 0. \tag{2.11}$$

In the ultra-relativistic case, all relations are obtained from the above through the substitution (1.6). The solutions of the first type represent turbulent waves. In Eq. (2.11), s denotes the velocity of sound: $s^{-2} = (\partial e/\partial p)_{\sigma/n}$.

3. BOUNDARY CONDITIONS AT THE SURFACE OF DISCONTINUITY

The continuity condition

$$[T_i'^f] = 0, \quad i = f,\ \tau,\ 0. \tag{3.1}$$

is satisfied at a normal discontinuity. The prime denotes a system of coordinates in which the disturbed discontinuity is at rest, f is the normal, and τ the tangent to the surface of discontinuity. In addition, we have in the relativistic case the following condition:

$$[n'f] = 0. \qquad (3.2)$$

The Rankine-Hugoniot equation, relating the thermodynamical variables at the discontinuity, is a consequence of Eqs. (3.1) and (3.2). It follows from the equation of state and the Rankine-Hugoniot equation that only one thermodynamical variable is independent at the discontinuity (we shall choose the pressure p). It should be noted that we deal with variables on one side of the surface of discontinuity only — behind the shock wave.

In the ultra-relativistic case it is clear from thermodynamical identities that there is only one independent thermodynamical variable at all. In both cases, therefore, we have at the discontinuity

$$\delta w = q \delta p, \qquad (3.3)$$

where, in the relativistic case

$$q = (\partial w / \partial p)_{\mathrm{H}} \qquad (3.4)$$

is the derivative along the Hugoniot adiabatic. In ultra-relativistic case, when there is no Hugoniot equation, but $\mu = 0$,

$$q = dw / dp. \qquad (3.5)$$

The system (3.1) and (3.3) is the complete system of boundary conditions at the perturbed discontinuity in linear approximation for both limiting cases.

We shall write down certain relations at the discontinuity, omitting the primes. From Eq. (3.1) we have

$$[wu_f^2] = -[p], \quad [wu_f u_\tau] = 0, \quad [wu_f u_0] = 0. \qquad (3.6)$$

It follows that the discontinuities of the components of the three-dimensional velocity are:

$$[u_\tau / u_0] = 0, \quad [u_f / u_0] = -[p] / wu_f u_0. \qquad (3.7)$$

Furthermore,

$$[wu_\tau^2] = \frac{u_\tau^2}{u_0^2}[wu_0^2], \quad [wu_0^2] = \frac{[w]-[p]}{1-u_\tau^2/u_0^2}; \qquad (3.8)$$

$$\left[\frac{u_0}{u_f}\right] = \frac{1}{wu_f^2}\frac{\bar{u}_0}{\bar{u}_f}[p], \qquad [wu_0^2] = \frac{u_0\bar{u}_0}{u_f\bar{u}_f}[p]; \qquad (3.9)$$

$$[e] = [w] - [p] = \left(1 - \frac{u_\tau^2}{u_0^2}\right)\frac{u_0\bar{u}_0}{u_f\bar{u}_f}[p]. \qquad (3.10)$$

The last relation yields for $u_\tau = 0$

$$[e]/[p] = u_0\bar{u}_0 / u_f\bar{u}_f. \qquad (3.11)$$

All the above relations are valid for the ultra-relativistic case as well. Making use of Eq. (3.2), $u_f = jV$, where $V = 1/n$ and $[j] = 0$, we obtain

$$j^2 = -[p]/[wV^2]. \qquad (3.12)$$

Since, according to (3.6) we have

$$[w^2u_f^2u_\tau^2] = [w^2u_f^2u_0^2] = 0,$$

then it follows that

$$j^2 = -[w^2V^2]/[w^2V^4]. \qquad (3.13)$$

Eliminating j^2 from Eqs. (3.12) and (3.13), we obtain the Rankine-Hugoniot equation:[4]

$$[p] = [w^2V^2] / (wV^2 + \bar{w}\bar{V}^2). \qquad (3.14)$$

4. DERIVATION OF THE CHARACTERISTIC EQUATION

In order to write down the perturbed boundary conditions it is necessary to carry out a transformation from the system (I) in which the unperturbed discontinuity surface is at rest, to the system (II) where the normal velocity of the perturbed discontinuity vanishes. The Lorentz formulae of transformation to the moving (primed) system (the motion along the i-axis with a 4-velocity U^i) are:

$$A_i' = U^0 A_i + U^i A_0, \quad A_0' = U^0 A_0 + U^i A_i, \qquad (4.1)$$

where there is no summation over double indices.

From the equation of the perturbed surface (2.1) we find the normal $f_\alpha(-ik\eta, 1, 0)$ and the tangent $\tau_\alpha(1, ik\eta, 0)$, $k \equiv k_1$. The velocity of the surface of discontinuity in (I) is

$$D_y = ik_0\eta, \quad D^0 = 1. \qquad (4.2)$$

The perturbed hydrodynamical velocities in (I) are

$$u_y + \delta u_y, \quad \delta u_x, \quad u_0 + \delta u_0; \quad \bar{u}_y, \ 0, \ \bar{u}_0. \qquad (4.3)$$

We next project on the directions of f and τ:

$$D_f = ik_0\eta; \quad D_\tau = 0; \quad u_f = u_y + \delta u_y; \quad \bar{u}_f = \bar{u}_y;$$
$$u_\tau = \delta u_x + ik\eta u_y; \quad \bar{u}_\tau = ik\eta\bar{u}_y.$$

By means of Eq. (4.1) we go over to system (II), assuming that $U_i = D_f = ik_0\eta$:

$$u_f' = u_y + \delta u_y + ik_0\eta u_0, \quad u_\tau' = \delta u_x + ik\eta u_y,$$
$$u_0' = u_0 + \delta u_0 + ik_0\eta u_y, \qquad (4.4)$$
$$\bar{u}_f' = \bar{u}_y + ik_0\eta\bar{u}_0, \quad \bar{u}_\tau' = ik\eta\bar{u}_y, \quad \bar{u}_0' = \bar{u}_0 + ik_0\eta\bar{u}_y.$$

It follows from $u_i u^i = -1$ that

$$u_y\delta u_y = u_0\delta u_0. \qquad (4.5)$$

We shall denote by ~ the perturbed variables

$$\tilde{w} = w + \delta w, \quad \tilde{\bar{w}} = \bar{w}, \quad \tilde{p} = p + \delta p, \quad \tilde{\bar{p}} = \bar{p}. \qquad (4.6)$$

Boundary conditions (3.1) become then

$$[\widetilde{w}u_f'^2] = -[\widetilde{p}], \quad [\widetilde{w}u_f'u_\tau'] = 0, \quad [\widetilde{w}u_f'u_n'] = 0,$$

substituting Eqs. (4.4) – (4.6) into the above and eliminating η, we find

$$\delta u_x = \frac{k}{k_0} P\,\delta p, \qquad P = \frac{1 + u_y^2(2-q)}{2u_0wu_y^2([e]/[p]-1)}; \tag{4.7}$$

$$\delta u_y = Q\delta p, \qquad Q = -(1+qu_y^2)/2wu_y. \tag{4.8}$$

In carrying out the transformations it is convenient to use the formulae of Sec. 3.

We shall now make use of the variational equations of motion

$$\delta u_i = \delta u_i^{(1)} + \delta u_i^{(2)}.$$

According to Eqs. (4.5) and (2.7), the first of conditions (2.8) becomes

$$ku_yu_0\delta u_x^{(1)} + k_0\delta u_y^{(1)} = 0.$$

Multiplying Eq. (4.7) by ku_yu_0 and Eq. (4.8) by k_0, adding them together, and then substituting $\delta u_i^{(2)}$ from Eq. (2.1) we obtain, since $\delta p \neq 0$

$$k^2k_0u_yu_0 + k_0^2(u_yk_lu^l + k_y) = -wk_lu^l(k^2u_yu_0P + k_0^2Q). \tag{4.9}$$

Equations (4.9) and (2.11) constitute a system of characteristic equations for k_0 and k_y as functions of k in both limits of relativistic hydrodynamics. The variables P and Q which are independent of k^l are given by Eqs. (4.7) and (4.8) and the parameter q by Eqs. (3.4) and (3.5) for the relativistic and ultra-relativistic cases respectively.

5. THE CHARACTERISTIC EQUATION

The dispersion relations obtained above describe, in the coordinate system I in which the medium as a whole moves with the 4-velocity u_i, the possible motions of the medium, due to perturbations of the boundary conditions at the discontinuity, taking place behind the shock wave. Let us now go over to a system III where the undisturbed medium behind the shock wave is at rest (we denote variables in III by $''$). We have

$$k_y = u^0k_y'' - k_0''u_y, \quad k_x \equiv k = k'', \quad k_0 = u^0k_0'' - k_y''u_y,$$

$$u_x'' = u_y'' = 0; \quad u_0'' = -1.$$

Let us put $k^{0''} = \Omega$. Then $k_\ell u^\ell = -\Omega$.

Equation (2.11) becomes in III

$$k^2 + k_y''^2 = \Omega^2/s^2. \tag{5.1}$$

Let us introduce polar coordinates

$$k'' = (\Omega/s)\sin\varphi; \quad k_y'' = (\Omega/s)\cos\varphi. \tag{5.2}$$

Equation (5.1) becomes then an identity. In system I we have

$$k_y = \Omega\left(\frac{u^0}{s}\cos\varphi + u_y\right), \qquad k = \frac{\Omega}{s}\sin\varphi,$$
$$k_0 = -\Omega\left(u^0 + \frac{u_y}{s}\cos\varphi\right). \tag{5.3}$$

Substituting into Eq. (4.9) we obtain the characteristic equation for $\cos\varphi$:

$$\cos^2\varphi\left\{\frac{u^{02}u_y}{s^2} - \frac{u_yu^0}{s^2}wP - \frac{u_y^2wQ}{s^2}\right\}$$
$$+\cos\varphi\left\{\frac{u^{03}}{s} + \frac{u^0u_y^2}{s^3} - 2\frac{u^0u_y}{s}wQ\right\} \tag{5.4}$$
$$+\left\{\frac{u_yu^{02}}{s^2} - u^{02}wQ + \frac{u_yu^0}{s^2}wP\right\} = 0.$$

Equation (5.4) corresponds, in non-relativistic hydrodynamics, to D'iakov's characteristic equation.[1] Like that equation, Eq. (5.4) is quadratic, which makes it possible to study it by the method of Ref. 1, as indicated by Landau.

6. INVESTIGATION OF THE CHARACTERISTIC EQUATION

Using Eq. (5.2), we can express Ω in terms of k (assumed to be real):

$$\Omega \doteq sk/\sin\varphi; \tag{6.1}$$

$$k_y = \frac{sk}{\sin\varphi}\left(\frac{u^0}{s}\cos\varphi + u_y\right), \quad k_0 = -\frac{sk}{\sin\varphi}\left(u^0 + \frac{u_y}{s}\cos\varphi\right). \tag{6.2}$$

The conditions of instability are

$$\operatorname{Im} k_0 < 0, \quad \operatorname{Im} k_y > 0. \tag{6.3}$$

Let us introduce real variables ρ and ψ

$$\cot(\varphi/2) = \rho e^{i\psi} \tag{6.4}$$

Using the inequalities

$$u^0/s - u_y > 0, \quad u^0 - u_y/s > 0,$$

we can write Eq. (6.3) in the following form:

$$|x| > 1, \text{ where } x = \frac{1+\cos\varphi}{1-\cos\varphi}\cdot\frac{u^0 + u_y/s}{u^0 - u_y/s}. \tag{6.5}$$

Since it is necessary for instability that k_0 and k_y were complex, then $\cos\varphi$ and $\sin\varphi$ cannot be real simultaneously. We have therefore

$$|\cos\varphi| > 1 \quad \text{for } \operatorname{Im}\cos\varphi = 0. \tag{6.6}$$

Rearranging Eq. (5.4) in terms of x we obtain

$$x^2(u^{02} - u_y^2/s^2)(u^0/s - wQ)$$
$$+2x\{2(u^0u_y/s^2)wP - wQ(u^{02} - u_y^2/s^2)\} \tag{6.7}$$
$$-(u^{02} - u_y^2/s^2)(u^0/s + wQ) = 0.$$

It is necessary to find such relations between the coefficients that inequality (6.5) is satisfied for at least one root of Eq. (6.7). Let us denote the roots by x_1 and x_2; we have then the following well-known inequalities for the quadratic equation $ax^2 + bx + c = 0$:

$$|a|>|c|, \quad |b|<|a+c|, \quad |x_1|, \quad |x_2|<1; \qquad (6.8)$$

$$|a|<|c|, \quad |b|<|a+c|, \quad |x_1|, \quad |x_2|>1; \qquad (6.9)$$

$$|a|\lesseqgtr|c|, \quad |b|>|a+c|, \quad |x_1|<1, \ |x_2|>1. \qquad (6.10)$$

It is easy to see that inequalities (6.9) are in contradiction with Eq. (6.7). The conditions of instability are therefore represented by (6.10) where x_1 and x_2 are real.

According to Eq. (6.5) $\cos\varphi$ is then real as well, i.e., relation (6.6) should be satisfied. This is possible only for $x < 0$. Consequently, for instability it is necessary that one root of Eq. (6.7) lies in the interval $(-\infty, -1)$. The condition for this is

$$a>0, \quad a-b+c<0; \qquad (6.11)$$

$$a<0, \quad a-b+c>0. \qquad (6.12)$$

We find $a - b + c$

$$a-b+c=-4u^0 u_y s^{-2} wP. \qquad (6.13)$$

It follows that for the regions of absolute instability

$$q>2+u_y^{-2}, \qquad (6.14)$$

or

$$q<-u_y^{-2}(1+2u^0 u_y/s). \qquad (6.15)$$

Relations (6.14) and (6.15) correspond, in non-relativistic hydrodynamics, to the regions of instability obtained by D'iakov.

Inequality (6.14) can be written in a more revealing form using the identity $q \equiv (\partial e/\partial p)_H + 1$

$$v^2>s_H^2, \text{ where } s_H^{-1}=\sqrt{\left(\frac{\partial e}{\partial p}\right)_H} \qquad (6.16)$$

is the "sound velocity" on the Hugoniot adiabatic.

In the ultra-relativistic case one should substitute in Eq. (6.7) s^2 for s_H^2 according to Eq. (3.5). Condition (6.16) becomes then identical with that of the instability region (1.7), that is, it does not represent an additional limitation. Inequality (6.15) cannot be satisfied since $q > 0$.

For ultra-relativistic shock waves, therefore, there is no region of absolute instability.

7. SPONTANEOUS EMISSION OF SOUND FROM THE DISCONTINUITY

An interesting result of the work of D'iakov is the discovery of regions in which the solution is of the form of undamped (in linear approximation) waves propagating from the discontinuity. We cannot exclude the possibility that, in spontaneous emission of sound, the energy of shock wave is fed into the emitted waves during such a long period that we have to consider the phenomenon as separate from the cases of stable and unstable motion.

We shall find the condition for spontaneous emission of sound from the discontinuity. In system (I) where the unperturbed discontinuity is at rest (and the fluid moves with velocity u_i), the velocity V_α of waves emitted from the discontinuity should have a positive normal component V_y (in contrast to waves incident upon the discontinuity, for which V_y is negative). The condition for emission is

$$\operatorname{Im} k^0=0, \quad \operatorname{Im} k_y=0, \quad V_y>0. \qquad (7.1)$$

Velocity V_α can be found, for example, by differentiating Eq. (2.11)

$$V_\alpha=\partial k^0/\partial k^\alpha, \quad \alpha=1, 2, 3. \qquad (7.2)$$

We obtain:

$$V_\alpha=\frac{k_i u^i u_\alpha(1-s^{-2})+k_\alpha}{k_i u^i u^0(1-s^{-2})+k^0}. \qquad (7.3)$$

Using Eq. (5.3) we shall express V_y through variables measured in system (III) where the fluid behind the discontinuity is at rest:

$$V_y=\frac{u_y/s+u^0\cos\varphi}{u^0/s+u_y\cos\varphi}. \qquad (7.4)$$

The condition of spontaneous emission of sound (7.1) becomes

$$-M<\cos\varphi<1, \qquad (7.5)$$

where M is the Mach number; $M = v/s = u_y/su^0$.

The waves emitted by the discontinuity can move, with respect to fluid at rest, in the direction opposite to the motion of the shock wave ($0 < \cos\varphi < 1$) as well as following it ($-M < \cos\varphi < 0$). In the latter case the emitted sound wave lags behind the shock wave, while continuing to propagate in the same direction as seen by observer situated in system (III).

Inequality (7.5) remains valid in non-relativistic hydrodynamics.[5] We shall find now by means of Sturm's theorem the condition that the roots of Eq. (5.4) lie outside the instability region (6.14) — (6.15) and that at least one of them satisfies the relation (7.5). For the region of spontaneous emission of sound we obtain the double inequality

$$-\frac{1}{u_y^2}\left(1+2u^0\frac{u_y}{s}\right)<q$$
$$<-\frac{1}{u_y^2}-\frac{1-M^2-(M^2u_y^2\varkappa)(1+2u_y^2)}{1-M^2+M^2/u_y^2\alpha} \tag{7.6}$$

where

$$\varkappa=[\omega]/[p]-2=(1-v\bar{v})/v\bar{v}.$$

In the ultra-relativistic case $q = 4$, $\alpha = 2$, $s^{-2} = 3$.

The left-hand side of inequality (7.6) is satisfied for all values of v. The right-hand side can be written in the following form:

$$v^4-{}^4/_9v^2+{}^1/_{27}>0,$$

which is satisfied by $v < 1/3$ and $v > 1/\sqrt{3}$. The latter case corresponds to instability and should be excluded. Making use of the relation $v\bar{v} = 1/3$ which follows for the ultra-relativistic case from Eq. (3.11) we obtain that for emission it is necessary that $\bar{v} > 1$, i.e., the velocity of propagation of shock waves in the part of the medium ahead of the front should be greater than velocity of light. The spontaneous emission of sound is therefore impossible in ultra-relativistic hydrodynamics and, when the relation (1.7) is satisfied, shock waves are absolutely stable.

In conclusion the author wishes to express his gratitude to I. M. Khalatnikov for suggesting the present work, and to I. M. Lifshitz, I. M. Khalatnikov, and V. L. German for valuable remarks in discussion.

Note added in proof (December 22, 1957). In an article published in November, 1957, Iordanskii[6] treats by a somewhat different method the stability of non-relativistic shock waves with respect to small perturbations of the fluid behind the wave. The regions of absolute instability found by him are identical with those of D'iakov. Since D'iakov [who required for the waves emitted by the discontinuity that $0 < \cos\varphi < 1$ instead of relation (7.5)] did not find explicitly the region of spontaneous sound emission, there is some discrepancy with results of Iordanskii, which disappears when the latter are compared with those of Ref. 5. Iordanskii explained the discrepancy by the fact that the perturbations considered by him were more general than perturbations of the surface of discontinuity only. It can be easily seen, however, that only the perturbations from the region of compression which were reflected by the shock wave can contribute to the solution (otherwise an unstable flow without shock waves would occur). If that is the case, however, perturbations can be always considered as originating at the shock wave, as was indeed assumed by D'iakov.

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SOVIET PHYSICS JETP VOLUME 34 (7), NUMBER 1 JULY, 1958

PHOTOPRODUCTION OF STRANGE PARTICLES ON PROTONS

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The photoproduction of strange particles on protons is considered for the case of hyperon spins of $1/2$ and $3/2$. The angular distribution of K mesons is derived. Comparison of these distributions with experiment permits one to determine the parity of the (KY)-system with respect to the nucleon.

THE photoproduction of strange particles is of great interest from the point of view of the hypothesis of the "minimum electromagnetic interaction"[1] connected with the conservation of strangeness in electromagnetic processes. In addition, the investigation of the photoproduction of strange particles may lead to conclusions concerning the spin of hyperons and the parity of the (YK) system.

The aim of this work is an investigation of the photoproduction of strange particles on protons with the emission of charged K mesons:

$$\gamma + p \to \Lambda^0 + K^+ \tag{1'}$$

$$\gamma + p \to \Sigma^0 + K^+ \tag{1''}$$

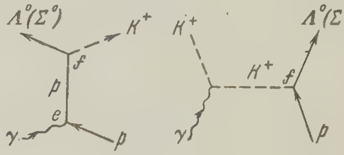
The cross sections of these processes is calculated by means of second order perturbation theory. The question of the applicability of perturbation theory to photoproduction of strange particles was considered recently by Gell-Mann,[2] who pointed out the possibility of applying weak coupling theory to the processes considered.

In accordance with experimental results,[3] we take the spin of the Λ^0-hyperon to be $1/2$ and that of the Σ^0-hyperon to be $3/2$.[3,4]

It is further assumed that the proton and the Λ^0 particle obey the Dirac equation (the interaction of the electromagnetic field with the magnetic moments of the particles is neglected), while the Σ-hyperon is described by the Rarita-Schwinger equation[5] for particles with spin $3/2$.

Both the direct interaction of the γ-quantum with the nucleon, and its interaction with the field of the virtual K-mesons, are considered.

To the extent that we consider processes of creation of K^+ mesons and that we assume the hypothesis of the conservation of strangeness in electromagnetic interactions, the only possible diagrams of the processes are the ones shown below in the figure.



a b

Carrying out the standard calculation, we obtain in the center-of-mass system (c.m.s.) the following expression for the angular distribution of K^+-mesons produced in reaction (1′) ($\hbar = c = 1$):

$$\frac{d\sigma}{d\omega} = \frac{e^2 f^2}{32\pi^2} \frac{q}{\varphi^4 \omega^3} \Big\{ mM (m^2 + \omega\varphi) + a\varphi (M (m^2 - \omega^2) - \omega^2 q \cos\theta) + \frac{2q^2\varphi^2}{(\varphi - M)^2 (1 - v\cos\theta)^2} [mM + a (M\sqrt{m^2 + \omega^2}) - \omega q \cos\theta] \sin^2\theta \Big\}, \tag{2}$$

$$\varphi = \omega + \sqrt{\omega^2 + m^2}, \quad q = \sqrt{(\varphi - M)^2 - \mu^2}.$$

Here v and q are the velocity and the momentum of the K meson in the c.m.s., ω is the energy of the incident γ-quanta, f is the coupling constant of the fields of the nucleons, hyperons and K mesons. The factor a is equal to +1 of to −1 depending upon whether the (hyperon — K^+ meson) system has the same parity as the proton or the opposite parity.

The calculation shows that the interaction of γ-quanta with the field of the virtual K^+ mesons gives an appreciably smaller contribution to the cross section than the direct interaction of a γ-quantum with a proton. From this it follows that if the ($\Lambda^0 K^+$) system has the same parity as the proton, the angular distribution of the K^+ mesons will be shifted in the c.m.s. towards the larger angles; for a parity of the ($\Lambda^0 K^+$) system opposite to the parity of the proton, the opposite is true.

For a 1-Bev γ-quantum (in the laboratory system) the order of magnitude of the total cross section of reaction (1′) is $f^2 \times 2 \times 10^{-32}$ cm^2.

Let us now consider the production of Σ-hyperons [reaction (1″)]. We proceed from the Rarita-Schwinger equation[5] for particles of spin $^3\!/_2$

$$(\gamma_k \partial/\partial x_k + M)\Psi_i = 0 \quad (i, k = 1, 2, 3, 4), \qquad (3)$$

where Ψ_i is a spin vector with the combined transformation properties of a four-vector and a Dirac bispinor. Ψ_i obeys the added conditions:

$$\partial\Psi_i/\partial x_i = 0, \quad \gamma_i\Psi_i = 0,$$

which lead to the fact that a direct interaction of the fields cannot be constructed. The only field couplings possible are those including the boson derivatives

$$H = (F/M)(\overline{\Psi}_i^{\Sigma}\Psi^N)\,\partial\varphi_K^+/\partial x_i + \text{complex conjugate},$$
$$H = (F/M^2)(\overline{\Psi}_i^{\Sigma}\gamma_m\Psi^N)\,\partial^2\varphi_K^+/\partial x_i\partial x_m + \text{complex conjugate}. \qquad (4)$$

Starting with the first interaction in (4), and using, for the spin- the projection operator of the Rarita-Schwinger equation in the form[6]

$$\Lambda_{il} = \sum_s \Psi_l^s\Psi_i^{s+} = \delta_{il} + \frac{2}{3M^2}p_ip_l - \frac{1}{3iM}(p_i\gamma_l - p_l\gamma_i) - \frac{1}{3}\gamma_i\gamma_l \qquad (5)$$

(p_i is a four-momentum) in the summation of the Σ^0-hyperon over the spin projections in the final state, we find that the angular distribution of K^+-mesons formed in reaction (1″) does not depend on the parity of the ($\Sigma^0 K^+$) system (relative to the proton). Further, the angular dependence of the cross section enters only in the term described by diagram b in Fig. 1. As in the previous case, the term corresponding to this diagram gives a small contribution to the cross section and can be neglected. Therefore the observed angular distribution must be isotropic in the c.m.s.

The total cross section of reaction (1″) has the form

$$\sigma = \frac{e^2}{6\pi}\left(\frac{F}{M}\right)^2 \frac{qmM(\varphi\omega + m^2)}{\omega^3\varphi^4}$$
$$\times\left[5 + 10\frac{\varphi}{M} + 14\left(\frac{\varphi}{M}\right)^2 - 6\left(\frac{\varphi}{M}\right)^3 + \left(\frac{\varphi}{M}\right)^4\right]. \qquad (6)$$

provided the square of the mass of the K meson is neglected in comparison with the square of the mass of the Σ hyperon.

Comparison of the obtained results with experiment will help decide the question of the parity of the ($\Lambda^0 K^+$) system relative to the proton. Let us mention further that if an isotropic distribution of K^+ mesons formed in reaction (1″) will be observed experimentally, then it may serve as an argument in favor of spin $^3\!/_2$ for the Σ hyperon.

Naturally, one must bear in mind at the same time that the results discussed above are obtained on the basis of perturbation theory, the applicability of which to the processes studied requires further investigation.

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SCATTERING OF LIGHT IN A FERMI LIQUID

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The scattering of light in a Fermi liquid is studied. An expression is obtained for the distribution of the scattered light with respect to angle and frequency, and estimates are carried out for liquid He^3.

LANDAU, in one of his papers,[1] has shown that it is possible for a special type of oscillation called "zero sound" to be propagated in Fermi liquids at sufficiently low temperatures. One of the conditions for the possibility of propagating "zero sound" is, in particular, the inequality

$$\omega_s \tau \gg 1, \tag{1}$$

where τ is a relaxation time which, for He^3, is of the order of $10^{-12}T^{-2}$ sec.[2] If, for example, the temperature where $\sim 0.01°$ K, then a frequency of more than 10^8 cycles per second would be required to observe zero sound, so that such an experiment would be very difficult to perform.

In place of this method, an indirect method can be proposed which consists in observing Rayleigh scattering of light in liquid He^3.* As is well known, in Rayleigh scattering there arise, in addition to the principal line, satellite lines differing from it in frequency by

$$\Delta\omega = \pm 2 \frac{u}{c} \omega \sin\frac{\theta}{2},$$

where u is the velocity of sound and θ is the scattering angle. The velocity of zero sound in He^3 is of the order of 2×10^4 cm/sec, so that $\Delta\omega \sim 10^{-6}\omega$.[2] Thus, in principle, the velocity of zero sound can be measured by observing the frequency distribution of the scattered light. The requirement (1) can be satisfied because of the high frequency corresponding to visible light.

Apart from the above considerations, the scattering of light in a Fermi liquid at sufficiently low temperatures has several specific features, which make it of interest to carry out a theoretical study of this phenomenon, particularly the distribution of intensity with respect to frequency.*

As is well known, the frequency dependence and the angular distribution for Rayleigh scattering of unpolarized light are given by the equation†

$$dh = \frac{\omega^4}{6\pi c^4}\frac{1}{2\pi V}\overline{\left|\int \delta D_{\Delta\omega}(\mathbf{r})\, e^{-i\mathbf{qr}}\, dV\right|^2}\,\frac{3}{4}(1+\cos^2\theta)\frac{d\Omega}{4\pi}\, d\Delta\omega, \tag{2}$$

where ω is the frequency of the incident light, θ is the scattering angle, $\mathbf{q}$ is the change in the wave vector of the light, equal in absolute magnitude to $(2\omega/c)\sin(\theta/2)$, and $\delta D_{\Delta\omega}$ is the Fourier component of the fluctuation of the dielectric permeability $\delta D(t)$:

$$\delta D_{\Delta\omega} = \frac{1}{\sqrt{t_0}}\int_0^{t_0} \delta D(t)\, e^{i\Delta\omega t}\, dt, \tag{3}$$

where t_0 is a certain large quantity which, in the final equation, will go to infinity.

The bar in equation (2) denotes averaging over the fluctuations. In what follows we shall, for simplicity, set the volume of the system equal to unity.

In view of the very small polarizability of helium atoms, it is possible to consider that the change in the dielectric permeability is due to density fluctuations, i.e., $\delta D = (\partial D/\partial N)\,\delta N$, where N is the number of particles per unit volume. However, according to the general theory of Fermi liquids set forth by Landau,[4] the number of excitations is equal to the number of atoms of the liquid. Consequently it is possible to write

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*The idea of using Rayleigh scattering was first suggested by S. P. Kapitza.

*We note that at high temperatures, where $\omega_s\tau \ll 1$, the scattering of light will be described by the usual equations.[3]

†Here dh is the so-called differential coefficient of extinction. The integral over dh with respect to $d\Omega$ and $d\Delta\omega$ gives the total coefficient of extinction h, representing the logarithmic decrement of the attenuation of the photon flux density in the medium.

$$\int \delta D_{\Delta\omega}(\mathbf{r})\, e^{-i\mathbf{qr}}\, dV = \frac{\partial D}{\partial N}\int \delta n_{\mathbf{q},\Delta\omega}(\mathbf{p})\, d\tau_p, \qquad (4)$$

$$d\tau_p = 2dp_x\, dp_y\, dp_z / (2\pi\hbar)^3,$$

where $\delta n_{\mathbf{q},\Delta\omega}(\mathbf{p})$ is the Fourier component with respect to $\mathbf{r}$ and t [the latter in the sense of Eq. (3)] of the fluctuation of the excitation distribution function.

Before proceeding with further calculations, we point out one important circumstance. In Eq. (2) the average is taken over all possible fluctuations. In the region of temperatures and frequencies where $\hbar\Delta\omega \gtrsim kT$, it is necessary to take account of quantum effects in the averaging process. This can be done satisfactorily, if one knows the result for the purely classical case ($kT \gg \hbar\Delta\omega$), by introducing a certain correction factor. For scattering in which there is an increase of the frequency by $\Delta\omega$ (anti-Stokes scattering), the factor is $(\hbar\Delta\omega/kT)N(\Delta\omega)$; for scattering in which the frequency decreases (Stokes scattering), the factor is $(\hbar\Delta\omega/kT)[N(\Delta\omega)+1]$, where $N(\Delta\omega)$ is the Bose distribution function. If a negative $\Delta\omega$ is used to describe Stokes scattering, it turns out that, because of the relation $N(-\Delta\omega)+1 = -N(\Delta\omega)$, the correction factor for both cases has the form

$$\frac{\hbar\Delta\omega}{kT}\left[e^{\hbar\Delta\omega/kT} - 1\right]^{-1}. \qquad (5)$$

We shall suppose thus that $kT \gg \hbar\omega$. To find the fluctuations of the distribution function we make use of the method suggested by Rytov[5] and by Landau and Lifshitz[6] for calculating fluctuations in electrodynamics and hydrodynamics.* In using this method we find the fluctuations of a "stray force" entering into the kinetic equation, whereupon by solving this equation we also obtain the fluctuations of the distribution function.

For the case of a Fermi liquid we will proceed from the kinetic equation, which we write in the form

$$\frac{\partial\delta n}{\partial t} + \frac{\partial\delta n}{\partial\mathbf{r}}\frac{\partial\varepsilon}{\partial\mathbf{p}} - \frac{\partial n_0}{\partial\mathbf{p}}\int f(\mathbf{p},\mathbf{p}')\,\frac{\partial\delta n(\mathbf{p}')}{\partial\mathbf{r}}\, d\tau_{p'}$$

$$= I(\delta n) + y(\mathbf{p},\mathbf{r},t). \qquad (6)$$

After the elimination of the "stray force" $y(\mathbf{p},\mathbf{r},t)$, this equation represents, in the approximation linear in δn, the kinetic equation for Fermi liquids found by Landau; here ϵ is the energy of an excitation, n_0 the equilibrium distribution function, and $f(\mathbf{p},\mathbf{p}')$ the function introduced in Ref. 4.

In what follows we shall be interested only in frequencies and temperatures for which Eq. (1) is satisfied, i.e., for which it is possible to ignore collisions. The detailed form of the collision integral will not be essential to us, since it plays the role of an auxiliary quantity in the calculations and in the final results can be set equal to zero. In view of this we set

$$I(\delta n) = -\delta n/\tau, \qquad (7)$$

where τ is a large quantity. It is next necessary to find the rate of change of the entropy. Remembering that the number of particles and the total energy are fixed and making use of the equation determining $f(\mathbf{p},\mathbf{p}')$,

$$\delta\varepsilon(\mathbf{p}) = \int f(\mathbf{p},\mathbf{p}')\,\delta n(\mathbf{p}')\, d\tau_{p'}, \qquad (8)$$

we find that

$$\dot{S} = -k\left\{\int \frac{\delta n\,[I(\delta n) + y]}{n_0(1-n_0)}\, d\tau_p\, dV\right.$$

$$\left. + \frac{1}{kT}\int f(\mathbf{p},\mathbf{p}')\,\delta(\mathbf{r}-\mathbf{r}')\,\delta n I(\delta n')\, d\tau_p\, dV\, d\tau_{p'}\, dV'\right\}. \qquad (9)$$

If we remember also that $n_0(1-n_0) \approx kT\delta(\epsilon-\mu)$, where μ is the chemical potential, it is not difficult to see that $\delta n(\mathbf{p})$ should have the form

$$\delta n(\mathbf{p}) = \delta n^{\varepsilon}(\vartheta,\varphi)\,\delta(\varepsilon-\mu), \qquad (10)$$

where ϑ and φ are the polar angles of the vector $\mathbf{p}$. This equation indicates that fluctuations of the distribution function take place only in the region of the Fermi surface.

It is natural to take the same form for y:

$$y(\mathbf{p}) = y^{\varepsilon}(\vartheta,\varphi)\,\delta(\varepsilon-\mu). \qquad (11)$$

We now introduce the notation

$$F(\gamma) = \left[f(\mathbf{p},\mathbf{p}')\frac{d\tau_{p'}}{d\varepsilon'}\right]_{\varepsilon=\varepsilon'=\mu},$$

where γ is the angle between $\mathbf{p}$ and $\mathbf{p}'$, and we expand δn, y, and F in spherical harmonics:

$$\delta n^{\varepsilon}(\vartheta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_n^m P_n^m(\cos\vartheta)\, e^{im\varphi},$$

$$y^{\varepsilon}(\vartheta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} y_n^m P_n^m(\cos\vartheta)\, e^{im\varphi}, \qquad (12)$$

$$F(\gamma) = \sum_{n=0}^{\infty} F_n P_n(\cos\gamma).$$

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*The authors are grateful to L. P. Gor'kov, I. E. Dzialoshinskii, and L. P. Pitaevskii for directing their attention to the possibility of applying this method to the kinetic equation.

Since δn^{ϵ} and y^{ϵ} are real quantities,

$$A_n^m = (A_n^{-m})^*, \; y_n^m = (y_n^{-m})^*.$$

Making use of Eq. (7) for the collision integral, we obtain the following equation for the rate of change of the entropy:

$$\dot{S} = \int dV \left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu} \frac{1}{T} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \left(\frac{F_n}{2n+1} + 1\right) \frac{1}{2n+1} \frac{(n+|m|)!}{(n-|m|)!} \left(\frac{A_n^m}{\tau} - y_n^m\right) A_n^{-m}. \tag{13}$$

We now introduce the notation

$$\dot{x}_n^m = -A_n^m/\tau + y_n^m. \tag{14}$$

Then if Eq. (13) is to have the form[6]

$$\dot{S} = -\sum_i X_i x_i$$

we have to take as the generalized force X_i the expression

$$X_n^m = \frac{1}{T}\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu} \left(\frac{F_n}{2n+1} + 1\right) \frac{1}{2n+1} \frac{(n+|m|)!}{(n-|m|)!} A_n^{-m}. \tag{15}$$

In Eq. (14) the quantity y_n^m plays the role of the "stray force." Writing this expression in the form

$$\dot{x}_n^m = -\sum_{n',\, m'} \gamma_{n,\, n'}^{m,\, m'} X_{n'}^{m'} + y_n^m,$$

where the coefficients γ are determined without difficulty from (15), we have, according to the general theory of fluctuations,

$$\begin{aligned} y_n^m(\mathbf{r}, t)\, y_{n'}^{m'}(\mathbf{r}', t') &= k\left(\gamma_{n,\, n'}^{m,\, m'} + \gamma_{n',\, n}^{m' m}\right) \delta(t - t')\, \delta(\mathbf{r} - \mathbf{r}') \\ &= \frac{2}{\tau} \delta_{n,\, n'} \delta_{m,-m'} \delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t') \\ &\times \left[kT\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu} \left(\frac{F_n}{2n+1} + 1\right) \frac{1}{2n+1} \frac{(n+|m|)!}{(n-|m|)!}\right]^{-1} \end{aligned} \tag{16}$$

Finally, having made use of (12) and the relation

$$\sum_n (2n+1)\, P_n(\cos\gamma) = 2\delta(\cos\gamma - 1),$$

we obtain after certain transformations the general expression

$$\begin{aligned} \overline{y(\mathbf{p}, \mathbf{r}, t)\, y(\mathbf{p}', \mathbf{r}', t')} &= \frac{2kT}{\tau} \delta(\mathbf{r} - \mathbf{r}')\, \delta(t - t') \\ &\times \left\{\frac{(2\pi\hbar)^3}{2} \delta(\mathbf{p} - \mathbf{p}')\, \delta(\varepsilon - \mu)\right. \\ &\left. + \delta(\varepsilon - \mu)\, \delta(\varepsilon' - \mu) \left(\frac{d\varepsilon}{d\tau_p}\right) \sum_{n=0}^{\infty} \frac{F_n P_n(\cos\gamma)}{1 + F_n/(2n+1)}\right\}. \end{aligned} \tag{17}$$

With the aid of this expression and the kinetic equation (6) we can calculate those fluctuations of the distribution function which are of interest to us. Since for the general case of an arbitrary function f this is a rather complex procedure, we restrict ourselves to the case f = const.*

Making use of the fact that the fluctuations occur only on the Fermi surface, and using Eq. (6), we express $\delta n_{\mathbf{q},\Delta\omega}^{\epsilon}(\vartheta, \varphi)$ in terms of the corresponding Fourier components of $y^{\epsilon}(\vartheta, \varphi)$. This gives

$$\begin{aligned} &\int \delta n_{\mathbf{q},\, \Delta\omega}^{\varepsilon}(\vartheta, \varphi) \frac{d\Omega}{4\pi} \\ &= \int \frac{y_{\mathbf{q},\, \Delta\omega}^{\varepsilon}(\vartheta, \varphi)\, (d\Omega/4\pi)}{-i\omega + 1/\tau + i\mathbf{q}\mathbf{v}} \Bigg/ \left(1 + F_0 \int \frac{i(\mathbf{q}\mathbf{v})\, (d\Omega/4\pi)}{-i\omega + 1/\tau + i\mathbf{q}\mathbf{v}}\right), \end{aligned} \tag{18}$$

where $\mathbf{v} = (\partial\epsilon/\partial\mathbf{p})_{\epsilon=\mu}$. Averaging the square of the absolute value of this expression with the help of Eq. (17) [keeping in mind that the Fourier component with respect to the time is derived according to Eq. (3)], we find

$$\begin{aligned} &\overline{\left|\int \delta n_{\mathbf{q},\, \Delta\omega}^{\varepsilon}(\vartheta, \varphi) \frac{d\Omega}{4\pi}\right|^2} = 2\left(\frac{d\varepsilon}{d\tau_p}\right)_{\varepsilon=\mu} \frac{kT}{\tau} \left[\frac{1}{2} \int_{-1}^{1} \frac{dx}{|qvx - \omega + i/\tau|^2}\right. \\ &\left. - \frac{F_0}{1 + F_0} \left|\frac{1}{2} \int_{-1}^{1} \frac{dx}{qvx - \omega + i/\tau}\right|^2\right] \Bigg/ \left|1 + \frac{F_0}{2} \int_{-1}^{1} \frac{qvx\, dx}{qvx - \omega + i/\tau}\right|^2 \end{aligned} \tag{19}$$

We are interested in the limiting value of this expression when $\tau \to \infty$. When $qv > |\Delta\omega|$, the denominator has no poles, and what turns out to be important is the residue of the integral in the numerator. For this we obtain

$$\begin{aligned} &\frac{1}{2\pi} \overline{|\delta n_{\mathbf{q}\Delta\omega}(\mathbf{p})\, d\tau_p|} \\ &= kT\left(\frac{d\tau_p}{d\varepsilon}\right) \frac{1}{qv} \left\{\left[1 + F_0\left(1 - \frac{\Delta\omega}{2qv} \ln \frac{qv + \Delta\omega}{qv - \Delta\omega}\right)\right]^2 + \left(\frac{F_0 \Delta\omega\pi}{2qv}\right)^2\right\}^{-1} \end{aligned} \tag{20}$$

In the opposite case, i.e., when $qv < |\Delta\omega|$, it is the pole in the denominator of (19) which is important. It is not difficult to see that such a pole arises if $F_0 > 0$ and the equality $\Delta\omega = \pm \eta qv$ holds, where η satisfies the equation

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*Actually, for He^3, the function f is not constant. It is possible to find the first two terms of the series (12) from the experimental data,[2] which give $F(\gamma) = 5.2 + 1.3 \cos\gamma$.

$$1+F_0\left[1-\frac{\eta}{2}\ln\left(\frac{\eta+1}{\eta-1}\right)\right]=0. \tag{21}$$

This expression agrees with Eq. (14) of Ref. 1 for the velocity of zero sound. Using the relation

$$\frac{1}{\pi}\lim_{\tau\to\infty}\frac{1/\tau}{(\omega-\omega_0)^2+\tau^{-2}}=\delta(\omega-\omega_0),$$

we obtain without difficulty

$$\frac{1}{2\pi}\left|\overline{\int\delta n_{\mathbf{q},\,\Delta\omega}(\mathbf{p})\,d\tau_p}\right|^2 = kT\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu}\frac{2(\eta^2-1)}{F_0(1+F_0-\eta^2)}\left[\delta(\Delta\omega-\eta qv)+\delta(\Delta\omega+\eta qv)\right]. \tag{22}$$

Thus the angular distribution and the frequency dependence of the scattered light are expressed by the following equation [in which the quantum factor (5) has been introduced]:

$$dh=\frac{\omega^4}{4\pi c^4}\left(\frac{\partial D}{\partial N}\right)^2\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu}(1+\cos^2\theta)\frac{\hbar\Delta\omega}{e^{\hbar\Delta\omega/kT}-1}$$
$$\times\left[\frac{\theta(qv-|\Delta\omega|)}{2qv}\Big/\left\{\left[1+F_0\left(1-\frac{\Delta\omega}{2qv}\ln\frac{qv+\Delta\omega}{qv-\Delta\omega}\right)\right]^2+\left(\frac{F_0\Delta\omega\pi}{2qv}\right)^2\right\}\right.$$
$$\left.+\frac{\eta^2-1}{F_0(1+F_0-\eta^2)}\left[\delta(\Delta\omega-\eta qv)+\delta(\Delta\omega+\eta qv)\right]\right]\frac{d\Omega}{4\pi}d\Delta\omega, \tag{23}$$

where

$$\theta(y)=\begin{cases}1 & y>0\\ 0 & y<0\end{cases}$$

The result obtained has a simple physical significance. As can be easily seen, the frequency spectrum consists of a central part $-qv<\Delta\omega<qv$ and two sharp lines at $\Delta\omega=\pm\eta qv$. The central part corresponds to the Doppler width of the principal line. A comparison of Eq. (1) with Eq. (14) of Ref. 1, concerning the oscillations of a Fermi liquid, shows that the secondary lines appear as satellites in Rayleigh scattering, arising in connection with the possibility of propagating zero sound ($\eta v=u$). The relation between the intensities of the central part and of the secondary satellites depends in general upon the scattering angle. In the limiting cases of high temperatures ($kT\gg\hbar\omega u/c$) and low temperatures ($kT\ll\hbar\omega\times u/c$), however, this expression does not depend on angle. It is possible to carry out a numerical estimate for He^3, using the well known parameters.[2] It turns out that, for high temperatures, the central part has about 20% of the total intensity, and the secondary lines about 40% each. At low temperatures the distribution will be cut off on the side of positive $\Delta\omega$ because of the quantum factor. In particular, of the two satellites there will remain only the Stokes line for $\Delta\omega=-uq$, with 90% of the intensity. Only 10% of the total intensity will appear in the central line.

The total scattering intensity is obtained by integrating (23) with respect to $d\Delta\omega$ and $d\Omega$. For high temperatures ($kT\gg\hbar\Delta\omega\sim\hbar\omega u/c$) it is equal to

$$h=\frac{\omega^4kT}{6\pi c^4}\left(\frac{\partial D}{\partial N}\right)^2\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu}J_1, \tag{24}$$

where J_1 is a numerical integral, equal to about 0.5 for He^3. In the quantum limit ($kT\ll\hbar\omega u/c$) we have

$$h=\frac{\hbar\omega^5v}{6\pi c^5}\left(\frac{\partial D}{\partial N}\right)^2\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu}J_2. \tag{25}$$

Here J_2 is a numerical integral which is equal to about 0.2 for He^3. In order to make a quantitative estimate of the above expressions for He^3, it is necessary to know the quantity $\partial D/\partial N$. There have been no measurements of this quantity; consequently we set $D-1$ proportional to N, and evaluate the coefficient of proportionality from data on liquid He^4. For a wavelength $\lambda=5461$ A the index of refraction of liquid He^4 is equal to 1.027. This gives $\partial D/\partial N=2.5\times10^{-24}$. Substitution in Eqs. (24) and (25) gives

$$\begin{aligned}h(\mathrm{He}^3)&\sim10^{-69}\,\omega^4T\ \mathrm{cm}^{-1}\ \text{for}\ \omega\ll2\cdot10^{17}\,T\ \mathrm{sec}^{-1},\\ h(\mathrm{He}^3)&\sim10^{-87}\,\omega^5\ \mathrm{cm}^{-1}\ \text{for}\ \omega\gg2\cdot10^{17}\,T\ \mathrm{sec}^{-1}.\end{aligned} \tag{26}$$

It is necessary to keep in mind that the frequency has to satisfy Eq. (1), i.e., $\Delta\omega\gg1/\tau$, or

$$\omega\gg10^{18}T^2\ \mathrm{sec}^{-1}. \tag{27}$$

If this condition were not satisfied, the line width would be too great. Thus for the visible region of frequencies, temperatures below 0.05° K are necessary. It is not difficult to see that to temperatures of the order of 0.01° K in the visible range of frequencies there will correspond a value of $\hbar(\mathrm{He}^3)\sim10^{-9}\ \mathrm{cm}^{-1}$, which certainly is too small for the effect to be measured.* Because of the fact that ω enters into the expression for h to a very high power, however, it is possible that the scattering can be successfully measured in the ultraviolet region.

In conclusion the authors express their gratitude to Academician L. D. Landau for his interest in the work.

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*For liquid He^4 in the visible region, $h\sim10^{-8}\ \mathrm{cm}^{-1}$ (Ref. 7). (Approximately the same result ought to be obtained for He^3 for $\Delta\omega\tau\ll1$.) Measurements of h have been made for He^4 (Refs. 8, 9), but they were at the limit of the experimental capabilities.

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THEORY OF DIFFUSE SCATTERING OF X-RAYS AND THERMAL NEUTRONS IN SOLID SOLUTIONS. III. ACCOUNT OF GEOMETRICAL DISTORTIONS OF THE LATTICE

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A method for treatment of various problems connected with static distortions of crystal lattices is proposed, in which the distortions are related to fluctuation waves of the composition and of the internal parameters. Scattering of x-rays and thermal neutrons in binary solutions of arbitrary composition and with arbitrary values of the short and long range order parameters is considered. Anisotropy of the crystal and its atomic structure are taken into account explicitly. The scattering intensity can be expressed in terms of the thermodynamical characteristics of the solution (or correlation parameters), elastic moduli (or interatomic interaction constants), and also in terms of parameters characterizing the dependence on the concentration of the cell shape and dimensions. The particular cases of ideal, dilute, almost completely ordered solutions and also of solutions located near the critical point on the decay curve or near phase-transition points of the second kind are investigated. The diffuse scattering intensity distribution in a Cu_3Au solution, calculated without making use of the theoretical parameters, agrees satisfactorily with the experimental distribution.

In earlier papers[1,2] (quoted in the following as I and II) the diffuse scattering of x-rays and neutrons in solid solutions was investigated within the framework of phenomenological[1] and microscopic[2] theories. In the course of these investigations it was assumed that the sole cause giving rise to diffuse scattering was the random distribution of the atoms among the lattice points of a geometrically ideal lattice. In this article we shall investigate the influence of geometrical distortions of the lattice, associated with the difference in the size of atoms of different kinds, on the scattering. During the last few years an intensive experimental study was begun of the diffuse scattering due to the above cause and of the weakening of the lines in an x-ray photograph (see, for example, Ref. 3 where references to other work are given). These problems have been studied theoretically for several particular cases in a number of papers.[4-9]

In this article we investigate the general case of substitution solutions of arbitrary composition with arbitrary values of the parameters of long and short range order without making use of the approximation of an elastic isotropic continuum. For the calculation of intensity we use a simple method, which is a generalization to the case when distortions are present of the method used in I and II. In this method, which can be called the method of fluctuation waves, we consider Fourier components of fluctuations of the composition, of the degree of long-range order, or of other internal parameters characterizing the state of the crystal. The equations which relate these fluctuation waves to waves of geometrical distortions turn out to be simple ones. This leads to the possibility of easily expressing the scattering intensity, which is related to the differences in the atomic factors and to the geometric distortions, in terms of the above Fourier components. The probability distribution of the fluctuations is obtained by means of the thermodynamic theory of fluctuations, as a result of which the intensity turns out to be expressed in terms of quantities determined from independent experiments. By means of this method the anisotropy of the crystal and its atomic structure have been taken into account explicitly. As in I and II, we have not taken into account in the present article Compton scattering, nor scattering by thermal vibrations, and the calculations are carried out within the framework of the kinematic scattering theory. The scattering of monochromatic radiation by a single crystal is considered.

1. DISORDERED SOLID SOLUTIONS

We consider a disordered solid substitution solution AB of arbitrary composition, and with arbitrary values of the parameters of short-range order, an elementary cell of which contains one atom. The distribution of the atoms A and B among the lattice points of the lattice is determined by assigning the numbers c_s, where s is the index of the lattice point, with $c_s = 1$ if an atom A occupies the s-th lattice point, and $c_s = 0$ if an atom B is situated there. The distribution of atoms can also be specified by giving not the c_s themselves, but their Fourier components defined by the relations:

$$c_s - c = \sum_{\mathbf{k}}{}' [c_{\mathbf{k}} \exp(-i\mathbf{k}\mathbf{R}_s) + c_{\mathbf{k}}^* \exp(i\mathbf{k}\mathbf{R}_s)];$$

$$c_{\mathbf{k}} = \frac{1}{N} \sum_s (c_s - c) \exp(i\mathbf{k}\mathbf{R}_s). \tag{1}$$

Here c is the atomic concentration of atoms A, $\mathbf{R}_s$ is the position vector of the s-th lattice point, the summation over $\mathbf{k}$ is taken over the vectors $\mathbf{k}/2\pi$ lying in one half of the first cell of the reciprocal lattice space and satisfying cyclic conditions.

The geometric distortions may be characterized by the displacements $\delta\mathbf{R}_s$ of atoms from the lattice points of an ideal periodic lattice having the same structure and the same lattice constant as the solution. To each distribution of atoms there corresponds a definite distribution of displacements. The $\delta\mathbf{R}_s$ may be represented in the form of a superposition of displacements corresponding to individual fluctuation waves. The discussion is considerably simplified if one assumes that the boundaries of the crystal are held by infinitely rigid walls and cannot be displaced. We denote by $\mathbf{k}'$ a vector parallel to the displacements of atoms in the $\mathbf{k}$-th wave and equal to $\mathbf{k}$ in absolute value. Then

$$\delta\mathbf{R}_s = \sum_k{}' \frac{a_{\mathbf{k}}\mathbf{k}'}{k^2} i [c_{\mathbf{k}} \exp(-i\mathbf{k}\mathbf{R}_s) - c_{\mathbf{k}}^* \exp(i\mathbf{k}\mathbf{R}_s)] \tag{2}$$

When $\mathbf{k} \to 0$ the $a_{\mathbf{k}}$ cease to depend on k (but do depend on the direction of $\mathbf{k}$). Explicit expressions for $a_{\mathbf{k}}$ are given below. Strictly speaking, we should also have taken into account in the expansion quadratic and higher-order terms in $c_{\mathbf{k}}$. Instead of this, in an alternative formulation of the problem, we could have taken into account fluctuations of short-range order parameters as well as fluctuations in composition in determining $\delta\mathbf{R}_s$, and could have also taken into account the fact that the quantities $a_{\mathbf{k}}$ should be evaluated for constant values of these parameters and may differ from the equilibrium values given below. The approximation adopted in the present article is valid if

$$|\partial v/\partial c| \gg |\partial^2 v/\partial c^2| \sqrt{\overline{c(1-c)}},$$

where v is the molar volume. This condition is fulfilled for solutions obeying Vegard's rule (linear dependence of v on c), and also for all solutions of low concentration of one of the components.

As is well known, the x-ray scattering intensity expressed in electronic units can be given in the form

$$I = \overline{\left|\sum_s f_s \exp(i\mathbf{q}_1, \mathbf{R}_s + \delta\mathbf{R}_s)\right|^2}. \tag{3}$$

Here $f_s = c_s f_A + (1 - c_s) f_B$ is the atomic scattering factor for an atom situated at the s-th lattice point, f_A and f_B are the atomic factors for atoms A and B, $\mathbf{q}_1$ is the difference between the wave vectors of the scattered and the incident waves, and the bar indicates averaging over all the permutations of the atoms among the lattice points.*

In calculating I by means of formula (3) it is convenient to break up f_s into its average value and its deviation from the average:

$$f_s = \bar{f} + (c_s - c)(f_A - f_B),\ \bar{f} = cf_A + (1-c)f_B.$$

The expression for I is then broken up into three terms each of which we shall evaluate separately:

$$\begin{aligned} I &= I_1 + I_2 + I_3 \\ &= |\bar{f}|^2 \sum_{s,\,s'=1}^{N} \exp(i\mathbf{q}, \mathbf{R}_s - \mathbf{R}_{s'}) \overline{\exp(i\mathbf{q}_1, \delta\mathbf{R}_s - \delta\mathbf{R}_{s'})} \\ &+ [\bar{f}(f_A^* - f_B^*) + \bar{f}^*(f_A - f_B)] \sum_{s,\,s'=1}^{N} \exp(i\mathbf{q}, \mathbf{R}_s - \mathbf{R}_{s'}) \\ &\times \overline{(c_s - c)\exp(i\mathbf{q}_1, \delta\mathbf{R}_s - \delta\mathbf{R}_{s'})} \\ &+ |f_A - f_B|^2 \sum_{s,\,s'=1} \exp(i\mathbf{q}, \mathbf{R}_s - \mathbf{R}_{s'}) \\ &\times \overline{(c_s - c)(c_{s'} - c)\exp(i\mathbf{q}_1, \delta\mathbf{R}_s - \delta\mathbf{R}_{s'})}. \end{aligned} \tag{4}$$

Here $\mathbf{q} = \mathbf{q}_1 - 2\pi\mathbf{K}_n$, where $\mathbf{K}_n$ is the vector of the reciprocal lattice lying in the same cell as the vector $\mathbf{q}_1/2\pi$ (so that $\mathbf{q}/2\pi$ lies in the first cell).

It can be seen from (4) that in order to evaluate I_1 it is necessary to find the statistical average of the expression $\exp(i\mathbf{q}_1, \delta\mathbf{R}_s - \delta\mathbf{R}_{s'})$. Such an average may be easily evaluated if one notes that in accordance with (2) this exponential breaks up into a product of factors corresponding to different $\mathbf{k}$ and adopts for the probability distribution of the quantities $c_\mathbf{k}$ the Gaussian distribution (cf. Refs. 11 and I):

$$w \sim \exp\Big(-\sum_\mathbf{k}{}' g_\mathbf{k} |c_\mathbf{k}|^2\Big). \tag{5}$$

(The formulas relating $g_\mathbf{k}$ to the thermodynamic characteristics of the solution are given below). Then

$$M(\mathbf{R}_s - \mathbf{R}_{s'}) = \overline{\exp(i\mathbf{q}_1, \delta\mathbf{R}_s - \delta\mathbf{R}_{s'})}$$

$$= \exp\Big[-2 \sum_\mathbf{k}{}' \frac{(\mathbf{q}_1\mathbf{k}')^2}{k^4 g_\mathbf{k}} a_\mathbf{k}^2 (1 - \cos \mathbf{k}, \mathbf{R}_s - \mathbf{R}_{s'})\Big]. \tag{6}$$

Substituting (6) into the first term of formula (4), factoring out e^{-L} where

$$L = 2 \sum_\mathbf{k}{}' k^{-4} (\mathbf{q}_1\mathbf{k}')^2 a_\mathbf{k}^2 / g_\mathbf{k}, \tag{7}$$

and expanding the exponential expression remaining in (6) into a series we obtain:

$$I_1 = 8\pi^3 \frac{N}{\Delta} |\bar{f}|^2 e^{-L} \delta(\mathbf{q}) + N^2 |\bar{f}|^2 e^{-L} \frac{(\mathbf{q}_1\mathbf{q}')^2}{q^4} \frac{a_\mathbf{q}^2}{g_\mathbf{q}}. \tag{8}$$

Here Δ is the volume of an elementary cell, the first term containing the δ function (which appears in the limiting case of an infinite crystal) determines the intensity of the regular Laue scattering, while the second term determines the intensity of the diffuse scattering.

In calculating I_1 terms in (8) are neglected which correspond to terms which appear in the expansion of the expression $M(\mathbf{R}_s - \mathbf{R}_{s'}) e^L$ and which contain products of two or more cosines. The discarded terms are much smaller than the term retained in (8) if the following condition holds

$$(Lq/k_m)\cos^{-2}(\mathbf{q}'\mathbf{q}_1) \ll 1, \tag{9}$$

where k_m is the maximum value of k lying in the first cell of the reciprocal lattice. In many cases condition (9) is satisfied for arbitrary $\mathbf{q}$, and it is always satisfied for sufficiently small $\mathbf{q}$ (with the exception of directions for which $\cos(\mathbf{q}'\mathbf{q}_1) = 0$). It is evident that if it should turn out to be necessary, it would not be difficult within the framework of the procedure outlined above to take into account also these discarded terms.

For the calculation of I_2 and I_3 one may make use of the relations obtained from (2), (5) and (6):

$$\overline{c_\mathbf{k} \exp(i\mathbf{q}_1, \delta\mathbf{R}_s - \delta\mathbf{R}_{s'})}$$

$$= (\mathbf{q}_1\mathbf{k}'/k^2 g_\mathbf{k}) a_\mathbf{k} [\exp(i\mathbf{k}\mathbf{R}_s) - \exp(i\mathbf{k}\mathbf{R}_{s'})] M(\mathbf{R}_s - \mathbf{R}_{s'}),$$

$$\overline{c_\mathbf{k}^* \exp(i\mathbf{q}_1, \delta\mathbf{R}_s - \delta\mathbf{R}_{s'})} = -(\mathbf{q}_1\mathbf{k}'/k^2 g_\mathbf{k}) a_\mathbf{k}$$

$$\times [\exp(-i\mathbf{k}\mathbf{R}_s) - \exp(-i\mathbf{k}\mathbf{R}_{s'})] M(\mathbf{R}_s - \mathbf{R}_{s'}). \tag{10}$$

Substituting (1), (5) and (10) into the second and third terms in (4) we obtain:

$$I_2 + I_3 = -N^2 [\bar{f}(f_A^* - f_B^*) + f^*(f_A - f_B)]$$

$$\times e^{-L} \frac{\mathbf{q}_1\mathbf{q}'}{q^2 g_\mathbf{q}} a_\mathbf{q} + N^2 |f_A - f_B|^2 e^{-L} \frac{1}{g_\mathbf{q}}. \tag{11}$$

In calculating I_2 and I_3 we have also neglected terms containing products of two or more cosines. However it may be shown that if condition (9) is satisfied for a given $\mathbf{q}$ then the neglected terms are much less than I_3 (if $L \ll 1$) or I_1 (if $L \gtrsim 1$).

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*To obtain the neutron scattering cross-section per unit solid angle one should replace f_A and f_B in the formulas for I by A_A and A_B, which are constants describing the interaction of the neutron with the nuclei (see Ref. 10), averaged over the isotopes and independent of the spin directions of the neutron and the nucleus, and one should multiply the result by $m^2/4\pi\hbar^2$, were m is the neutron mass.

It follows from (4), (8), and (11) that the expression for the intensity of scattering by a disordered solution may be written in the form:

$$I = 8\pi^3 \frac{N}{\Delta} |\bar{f}|^2 e^{-L} \delta(\mathbf{q}) + N^2 \frac{e^{-L}}{g_\mathbf{q}} \left| \bar{f} a_\mathbf{q} \frac{\mathbf{q}_1 \mathbf{q}'}{q^2} - f_A + f_B \right|^2. \quad (12)$$

In order to be able to apply formula (12) we must find expressions for the quantities $a_\mathbf{q}$ and $g_\mathbf{q}$. We consider first the quantities $a_\mathbf{q}$ which characterize the degree of deformation associated with the **q**-th fluctuation wave. For this purpose we shall make use of the conditions of equilibrium with respect to the displacement of the atoms of the solution. In the approximation of an elastic isotropic continuum the expression for the density F of the free energy of a solution, in which there is a non-uniform distribution of concentration $\delta c(\mathbf{r})$, may be written in the form (cf. Ref. 12, part II, Ch. 1):

$$F = F_0 - \varkappa K \delta c u_{ll} + \mu (u_{ij} - {}^1/_3 \delta_{ij} u_{ll})^2 + {}^1/_2 K u_{ll}^2,$$

where i, j, and ℓ denote Cartesian coordinates, u_{ij} is the deformation tensor, F_0 does not depend on u_{ij} and δc, k is the bulk modulus of elasticity, μ is the shear modulus, δ_{ij} is the Kronecker delta. In this and in the following expressions it is understood that summation is to be carried out over repeated indices. From this we obtain the following equation for the components of the stress tensor:

$$\sigma_{ij} = \partial F / \partial u_{ij} = -\varkappa K \delta c \delta_{ij} + K u_{ll} \delta_{ij} + 2\mu (u_{ij} - {}^1/_3 \delta_{ij} u_{ll}). \quad (13)$$

In the case when δc is constant along the sample and $\sigma_{ij} = 0$, Eq. (13) enables us to relate κ to the equilibrium change in the volume as the composition is changed. In this case $u_{ij} \sim \delta_{ij}$, and setting (13) equal to zero we find that $\kappa = u_{\ell\ell}/\delta c = (1/v)\, \partial v/\partial c$. If δc is a wave of fluctuations of composition along the x axis, then in planes perpendicular to this axis the displacements are equal to zero (the crystal boundaries do not move) and all the u_{ij} with the exception of u_{xx} are equal to zero. Substituting in (13) $c_\mathbf{k} \exp(-i\mathbf{k}\mathbf{R}_s)$ in place of δc, and the expression $a_\mathbf{k} c_\mathbf{k} \exp(-i\mathbf{k}\mathbf{R}_s)$ obtained from (2) in place of u_{xx}, setting σ_{xx} equal to zero and making use of the well known relations between k, μ and the Poisson ratio σ, we obtain:

$$a_\mathrm{k} = \frac{1+\sigma}{3(1-\sigma)} \varkappa = \frac{1+\sigma}{3(1-\sigma)} \frac{1}{v} \frac{\partial v}{\partial c}; \quad \mathrm{k}' = \mathrm{k}. \quad (14)$$

Thus, in the approximation of an isotropic continuum, $a_\mathbf{k}$ does not depend on **k**.

In the case of a single crystal

$$F = F_0 - \lambda_{ijlm} k_{lm} u_{ij} \delta c + {}^1/_2 \lambda_{ijlm} u_{ij} u_{lm},$$

where $\lambda_{ij\ell m}$ is the elasticity tensor. In order to determine $k_{\ell m}$ we may, as above, consider the case when $\delta c(\mathbf{r}) = \mathrm{const}$ and $\sigma_{ij} = 0$. The equation obtained from this shows that the $k_{\ell m}$ are equal to the result of dividing by δc the components of the deformation tensor $u_{\ell m}$ which arises as a result of changing the composition by δc. Evidently the tensor $k_{\ell m}$ is completely determined if we know the dependence on the composition of the dimensions and the shape of the ideal elementary cell. If the fluctuation wave is directed along the X′ axis with the unit vector $\mathbf{n} = \mathbf{k}/k$, then in the system of coordinates x′y′z′, only $u_{x'x'} \neq 0$, $u_{x'y'} \neq 0$, $u_{x'z'} \neq 0$. The quantities σ_{ij} are expressed in terms of the $u_{\ell m}$ by means of the formula

$$\sigma_{ij} = -\lambda_{ijlm} k_{lm} \delta c + \lambda_{ijlm} u_{lm}. \quad (15)$$

Replacing δc and $u_{x'x'}$ by periodic functions and taking into account that $\partial\sigma_{ij}/\partial x_j = 0$ we obtain from (15), (1), and (2) three equations for the $a_\mathbf{k} n_i'$, where $n' = \mathbf{k}/k$:

$$\lambda_{ijlm} n_j n_l a_k n'_m = \lambda_{ijlm} k_{lm} n_j. \quad (16)$$

Thus the $a_\mathbf{k}$ may be determined if we know the moduli of elasticity and the dependence on the composition of the dimensions and the shape of the elementary cell. The expressions for $a_\mathbf{k}$ become simplified for various specific structures. Thus in the case of cubic crystals

$$k_{lm} = \frac{1}{3} \frac{1}{v} \frac{\partial v}{\partial c} \delta_{lm}$$

and there exist only three different elastic moduli

$$\lambda_{xxxx} = c_{11}, \ \lambda_{xxyy} = c_{12} \ \text{И} \ \lambda_{xyxy} = c_{44}.$$

In this case, for directions of the type (100), expression (16) takes on the following form:

$$a_\mathrm{k} = \frac{1}{3} \frac{1}{v} \frac{\partial v}{\partial c} \frac{c_{11} + 2c_{12}}{c_{11}}. \quad (17)$$

In determining the quantities $a_\mathbf{k}$ for short wavelengths, the continuum approximation may no longer be used, and the atomic structure of the crystal must be taken into account explicitly. In the approximation under discussion, where the relation between $\delta\mathbf{R}_s$ and $c_\mathbf{k}$ is assumed to be linear, the expression for the free energy of the solution considered as a function of the static displacements of the atoms (positions of equilibrium for their vibrations) takes on the following form:

$$F' = F_0' - w_{ss'i} (c_{s'} - c) \delta R_{si} + {}^1/_2 V_{ss'ij} \delta R_{si} \delta R_{s'j}, \quad (18)$$

where the coefficients $w_{ss'i}$ and $V_{ss'ij}$ depend on the relative position of the lattice points s and

s', on the temperature and on c. The force $\mathbf{F}_s$ which acts on the s-th atom due to all the other atoms is equal to:

$$F_{si} = -\partial F' / \partial\delta R_{si} = -w_{ss'i}(c_{s'} - c) + V_{ss'ij}\,\delta R_{s'j}. \tag{19}$$

Substituting into (19) in place of $c_{s'} - c_s$ and $\delta R_{s'j}$ the partial ($\mathbf{k}$-th) waves appearing in the expansions (1) and (2), and equating to zero the components of the vectors $\mathbf{F}_s$, we obtain:

$$\begin{aligned} & a_{\mathbf{k}} n'_j V_{ss'ij} \exp(i\mathbf{k},\ \mathbf{R}_s - \mathbf{R}_{s'}) \\ &= (k/i)\, w_{ss'i} \exp(i\mathbf{k},\ \mathbf{R}_s - \mathbf{R}_{s'}) \quad (i = 1, 2, 3). \end{aligned} \tag{20}$$

Restricting ourselves to the few nonvanishing constants $V_{ss'ij}$ and $w_{ss'i}$, we can express these constants in terms of the moduli of elasticity and of quantities which characterize the dependence of the dimensions and the shape of the cell on the concentration. Subsequently the $a_{\mathbf{k}}$ will be determined for arbitrary $\mathbf{k}$.

We now go on to the determination of $g_{\mathbf{k}}$ and to the investigation of the expressions obtained for I in various special cases. In accordance with (5), $1/g_{\mathbf{k}} = \overline{|c_{\mathbf{k}}|^2}$. For ideal and for weak solutions (when the atoms are distributed among the lattice points at random) these quantities may be determined either directly from formula (1) (taking into account that $\overline{c_s^2} = \overline{c_s} = c$, $\overline{c_s c_{s'}} = 0$ for $s \neq s'$) or with the aid of the usual formulas of the thermodynamic theory of fluctuations (cf. I):

$$1/g_{\mathbf{k}} = c(1-c)/N.$$

Substituting this expression into (12) we shall find that the intensity of diffuse scattering I_φ by ideal solutions is equal to:

$$I_\varphi = Nc(1-c)e^{-L}\,|a_{\mathbf{q}}\bar{f}\mathbf{q}_1\mathbf{q}'/q^2 - f_A + f_B|^2, \tag{21}$$

where the $a_{\mathbf{q}}$ in various special cases are determined by formulas (14), (16), (17), and (20). For $a_{\mathbf{q}} = 0$ formula (21), and also the formulas given below for non-ideal solutions, reduce to the corresponding formulas of Refs. I and II. Therefore in solutions with components having nearly equal atomic radii, and not too closely equal atomic factors (for example in Ag — Au) one may use formulas given in I and II for all values of $\mathbf{q}$, with the exception of values lying in the immediate neighborhood of lattice points of the reciprocal lattice.

In the case of non-ideal solutions, when the correlation between the occupation of different lattice points by atoms of different kinds is significant, we have in accordance with (1):

$$1/g_{\mathbf{k}} = \overline{|c_{\mathbf{k}}|^2} = \frac{1}{N}\Big[c(1-c) - \sum_{\rho\neq 0} \varepsilon(\rho)\cos\boldsymbol{\rho}\mathbf{k}\Big]. \tag{22}$$

Here $\boldsymbol{\rho} = \mathbf{R}_{s'} - \mathbf{R}_s$ take on the values of all the position vectors of the lattice, and the correlation parameters are

$$\varepsilon(\rho) = -(\overline{c_s c_{s'}^{(\rho)}} - c^2),$$

where the averaging is carried out over all pairs of lattice points separated from one another by the distance $\boldsymbol{\rho}$. In accordance with (12) and (22), I_φ is in this case equal to:

$$\begin{aligned} & I_\varphi \\ &= N\Big[c(1-c) - \sum_{\rho\neq 0} \varepsilon(\rho)\cos\overline{\mathbf{q}\boldsymbol{\rho}}\Big] e^{-L}\,|a_{\mathbf{q}}\bar{f}\mathbf{q}_1\mathbf{q}'/q^2 - f_A + f_B|^2. \end{aligned} \tag{23}$$

Formula (23) is a generalization of the well-known formula (see, for example, Ref. 13) which expresses I_φ in terms of c and $\epsilon(\boldsymbol{\rho})$ to the case when geometrical distortions are present. This formula also enables us to determine $\epsilon(\boldsymbol{\rho})$ from the experimentally obtained distribution of intensity of diffuse scattering taking geometrical distortions into account. For this one has to carry out the Fourier transformation of I_φ divided by the last factor of formula (23). As a consequence of this the expressions for $\epsilon(\boldsymbol{\rho})$ can be obtained from the corresponding expressions of Ref. 13 by making the following replacements in the latter:

$$f_A - f_B \to f_A - f_B - a_{\mathbf{q}}\bar{f}\mathbf{q}_1\mathbf{q}'/q^2, \tag{24}$$

i.e.,

$$\varepsilon(\rho) = -\frac{1}{2}\int d\mathbf{k}\,\frac{I_\Phi}{N}\left|a_{\mathbf{k}}\bar{f}\,\frac{\mathbf{q}_1\mathbf{k}'}{k^2} - f_A + f_B\right|^{-2}\cos\mathbf{k}\boldsymbol{\rho}, \tag{25}$$

where in the course of integration $\mathbf{k}/2\pi$ take on values which lie inside the volume of an arbitrary elementary cell of the reciprocal lattice.

As was shown in II, certain thermodynamic constants (for example the energies of ordering) can be obtained directly from the experimentally-determined distribution of diffuse scattering. The inclusion of geometrical distortions in this method will also reduce to carrying out the replacement (24) in formulas (23) of II. The correction for the distortions is particularly important in the case of decaying solutions with appreciably different atomic radii. It should be noted that in the case of weak dependence of v on c and of not too similar atomic scattering factors ($a_{\mathbf{q}}|\bar{f}| < 0.1\,|f_A - f_B|$) the correction due to the distortions in the formulas for $\epsilon(\boldsymbol{\rho})$ and for the energies of ordering is significant only for small q ($q \ll k_m$) when the macroscopic formulas (16) and (17) can be used for $a_{\mathbf{q}}$. In the more general case when the correction is significant also for large q, $a_{\mathbf{q}}$ should be found with the aid of formula (20).

If the constants characterizing the thermodynamic properties of the solution, e.g. the energies of ordering, are known, then I_φ may be calculated with the aid of the formulas of II upon making the replacement (24). In the general case, the determination of I_φ by this method will not be sufficiently accurate, since it is based on a simplified statistical model of the solution. However, in the neighborhood of the lattice points of the reciprocal lattice (where the distortions play the most significant role), the evaluation of I_φ from the thermodynamic data can be carried out quite accurately. In this case [compare (9) of I with (5) of this article] $g_\mathbf{k}$ is determined for cubic crystals by means of the following formula:

$$g_\mathbf{k} = (V / kT)(\varphi_{cc} + \beta k^2). \qquad (26)$$

Here V is the volume of the crystal, $\varphi_{cc} = d^2\varphi/dc^2$, φ is the thermodynamic potential per unit volume, the derivative with respect to c is calculated for a given number of atoms (and not for a given volume), and β is of the order of magnitude of $(kT/\Delta)\, r_0^2$, where r_0 is the interatomic distance. Substituting (26) into (12) we find that for $q \ll k_m$

$$I_\Phi = \frac{N}{\Delta} \frac{kTe^{-L}}{\varphi_{cc} + \beta q^2} \left| a_\mathbf{q} \bar{f} \frac{\mathbf{q}_1 \mathbf{q}'}{q^2} - f_A + f_B \right|^2. \qquad (27)$$

In the general case of crystals of non cubic symmetry, βq^2 is replaced by the sum $\beta_{ij} q_i q_j$. If the dependence on concentration of the activity a_A of atoms A in solution AB is known, then φ_{cc} may be found from the formula:

$$\varphi_{cc} = \frac{kT}{(1-c)\Delta} \frac{1}{a_A} \frac{da_A}{dc}.$$

Thus, for $\mathbf{q} \to 0$, I may be calculated with the aid of parameters obtained from independent experimental data on the dependence of a_A on c, on the moduli of elasticity, and on the dependence on concentration of the dimensions and shape of the elementary cell. Within a certain region $q \ll k_m$, in accordance with (27), I_Φ depends only on the single unknown parameter β. It is evident that formula (27) also provides a method for an independent determination of activities from experimental data obtained from x-ray photographs.

Geometrical distortions are particularly significant if the molar volume depends strongly on c ($a_\mathbf{q}$ is large), and if f_A and f_B are nearly equal. Even if $a_\mathbf{q} \ll 1$ and $|f_A - f_B| \ll |\bar{f}|$, still for sufficiently small q, i.e., in the vicinity of the lattice points of the reciprocal lattice, the first term in the binomials of formulas (21), (23), or (27), which is determined by the distortions, becomes dominant. In this case, just as in the case of weak solutions discussed by Huang,[5] I_Φ tends to infinity like q^{-2} when a lattice point of the reciprocal lattice is approached. The coefficient of q^{-2} is proportional to $q_1^2 = (4\pi\lambda^{-1}\sin\theta)^2$, where λ is the wavelength and 2θ is the scattering angle. Therefore, the intensity, divided by $|f_A - f_B|^2$, of the background associated with distortions, in contrast to the intensity of the background associated with a difference in the atomic factors, does not recur periodically in the different cells of the reciprocal lattice, but grows with an increase in θ and with a decrease in λ (i.e., with an increase in the cell index). For small θ, when $\mathbf{q}_1 = \mathbf{q}$, I_Φ in accordance with (27) tends to a constant limit and does not vary proportionally to q^{-2}. The term in I_Φ which is proportional to q^{-2} contains $\cos^2\varphi$ as a factor where φ is the angle between $\mathbf{q}'$ and $\mathbf{q}_1$. Therefore, for small q, I_Φ depends appreciably not only on the distance to the nearest lattice point of the reciprocal lattice, but also on the mutual orientation of the vectors $\mathbf{q}'$ and $\mathbf{q}_1$. Since $a_\mathbf{q}$ also changes appreciably as the orientation of $\mathbf{q}$ with respect to the crystal axes is varied, the above behavior of I_Φ is determined not only by the factor $\cos^2\varphi$, but also by the factor $a_\mathbf{q}^2$, and will be different in the vicinity of the different lattice points of the reciprocal lattice. It should be noted that the small terms neglected in the derivation of formulas (8) and (11) have a different dependence on the direction of $\mathbf{q}$ than the terms which are retained; in particular, they do not vanish when $\cos\varphi = 0$. Therefore the characteristic form of the isodiffusion curves will be somewhat altered if we take into account the neglected terms, whose relative importance in accordance with (9) increases with increasing q and becomes insignificant as $q \to 0$. The cross products obtained on squaring the binomials in formulas (21), (23), and (27) change sign when $\mathbf{q}$ is replaced by $-\mathbf{q}$ and lead to an asymmetry in the distribution of intensity of diffuse scattering with respect to the lattice points of the reciprocal lattice.

It can be seen from formula (21) that, even in the case of ideal solutions, the dependence of I_Φ on concentration is, generally speaking, not simple, since it is determined by the factor $c(1-c)$, by the linear dependence of $\bar{f}$ on c, and also by the dependence on the concentration of $a_\mathbf{q}$ which is different for different solutions. If the correlation in the solution is significant, then the dependence of I_Φ on the concentration, as may be seen from (23) and (27), becomes even more complicated, owing to the fact that $\epsilon(\boldsymbol{\rho})$ and φ_{cc} depend strongly on c. Moreover, I_Φ also depends strongly on the annealing temperature T′ at which short range order is established. Apparently, for small q, a typical behavior will be an increase of I_ϕ with diminishing T' for decaying solutions, and an increase of I_ϕ for solutions that are becoming ordered (although exceptions are possible).

The intensity of diffuse scattering in the vicinity of lattice points of the reciprocal lattice becomes particularly great for solutions which are in the neighborhood of a critical point on the decay curve (point K in Fig. 2 of I). As is well known, $\varphi_{cc} = 0$ at a critical point. Therefore, right at the critical point, I_ϕ increases for small q proportionally to q^{-4} since $\beta q^2 \ll 1$ ($\beta k_m^2 \sim 1$) for $q \ll k_m$ and in this case I_ϕ is considerably larger than, say, for ideal solutions.* In the neighborhood of the critical point φ_{cc} is very small and for small q, when $\beta q^2 < \varphi_{cc}$, I_ϕ is proportional to q^{-2}, but with a very large coefficient of proportionality.†

If the sizes of the atoms A and B are nearly equal and $a_\mathbf{q}$ is small, then the dependence of I_ϕ on **q** agrees with that obtained in I for not too small q, when $|a_\mathbf{q} f q_1/q| \ll |f_A - f_B|$ (but $q \ll k_m$).

In order to calculate L, which determines the reduction in the intensity of the regular reflections, it is necessary in the general case, as follows from (7), to know $a_\mathbf{k}$ and $g_\mathbf{k}$ not only for long, but also for short wavelengths, i.e., it is necessary to utilize a specific atomic model. Such calculations will be carried out elsewhere. Here we shall merely quote the formula for L obtained by Huang[5] for weak solutions (and also applicable to ideal solutions) with a face-centered cubic lattice and improved in accordance with formula (14) by the addition of the factor $(1+\sigma)^2/9(1-\sigma)^2$, which takes into account the difference in the displacements brought about by a point defect in an infinite and a finite crystal (cf. Ref. 14):

$$L = 2B\frac{\sin^2\theta}{\lambda^2} = \frac{16\pi^2}{\lambda^2}u_i^2 n_i^2$$

$$= 0.234c(1-c)\left(\frac{1+\sigma}{1-\sigma}\right)^2\left(\frac{1}{v}\frac{\partial v}{\partial c}\right)^2\frac{d_0^2}{\lambda^2}\sin^2\theta. \qquad (28)$$

Here u_i^2 is the mean square of the i-th component of the static displacements of the atom, and d_0 is the length of an edge of the cubic cell. In deriving formula (28) atomic structure was taken into account explicitly, but the displacements of each atom were calculated in the approximation of an isotropic continuum and may differ considerably from the true displacements. In the case of non-ideal solutions L, generally speaking, decreases when short-range order, which is characteristic of decaying solutions, is established.

The evaluation of L can be carried out without taking atomic structure into account if the solution is near a critical point on the decay curve. Since, in accordance with (26), $1/g_\mathbf{k}$ decreases rapidly in this case (maximum value at $\mathbf{k} = 0$), the region of small **k** is the most important one in the summation (7). Making the transition in (7) from summation to integration over **k**, extending the region of integration to infinity, and making use of (14), we find in the approximation of the isotropic continuum that near the critical point

$$L = \frac{8\pi}{27}\left(\frac{1+\sigma}{1-\sigma}\right)^2\frac{kT}{\sqrt{\varphi_{cc}\beta}}\frac{1}{\lambda^2}\sin^2\theta.$$

In the case of specific crystals, it is not difficult to obtain L near a critical point, even without using the approximation of an isotropic continuum, by making use of (16) or (17) and by carrying out numerical integration over the angles. The above formula allows us to calculate L if the value of β has been previously determined from the data on diffuse scattering. As the critical point is approached the intensity of regular reflection is reduced in proportion to

$$\exp\left(-\text{const}\,\varphi_{cc}^{-1/2}\right).$$

This law ceases to hold in the immediate neighborhood of the critical point $L > 1$ and the stricter criterion (9) (see note†) is no longer satisfied.

It follows from (7), (16), and (20) that for the same $\sin\theta/\lambda$ (same q_1), but for different directions of $\mathbf{q}_1$, the values of L for non-cubic crystals may differ appreciably. Formula (28), while giving a correct estimate of L, may, generally speaking, lead to appreciable quantitative error. An evaluation of L, taking atomic structure into account by means of formulas (7) and (20), yields an improved expression for L and permits us to investigate its dependence on the direction of $\mathbf{q}_1$ and on the correlation parameters.

Expression (27) for I_ϕ in the vicinity of regular reflections was compared with experimental data by Borie[9] for a disordered solution Cu_3Au quenched at 600°C. The value of $a_\mathbf{q}$ for **q** directed parallel to a cubic axis is computed by means of formula (17). In doing this it was assumed in accordance with Ageev and Shoikhet[15] that $v^{-1}\partial v/\partial c = 0.4$, and in accordance with the data of Siegel[16] (for ~ 400°C.) it was assumed that $(c_{11} + 2c_{12})/3c_{11} = 0.87$. With the aid of thermodynamic data on the activity of Cu in Cu — Au (Ref. 17) it was found that at 600°C. $\varphi_{cc} = 13kT/\Delta$. For small q it was possible to neglect βq^2 in comparison with φ_{cc}. For the ratio of the atomic factors corresponding to the reflection (200), the value $f_{Au}/f_{Cu} = 3.0$ was adopted. According to Borie,[9] the factor e^{-L} for the reflection under discussion practically does not differ from unity (~0.96). With the aid of these values for the parameters, we have calculated by means of formula (27) the theoretical curve of the dependence of I_ϕ/Nf_{Cu}^2 on q/q_1 for the direction (100) in the reciprocal lattice in the vicinity of the (200) reflection (the solid curve in the figure). In the same figure we have also shown the experimental curve (dotted line) obtained by Borie[9] after subtracting the Compton and the thermal scattering. The difference between the calculated and the measured values of I_ϕ lies within the limits of error of the experimental determination of the parameters of the theory (the inaccuracy in the determination of φ_{cc} may be particularly significant). Therefore the agreement between theory and experiment may be considered to be good, particularly if we take into account the fact that in constructing the theoretical curve all the parameters have been determined from independent experimental data.

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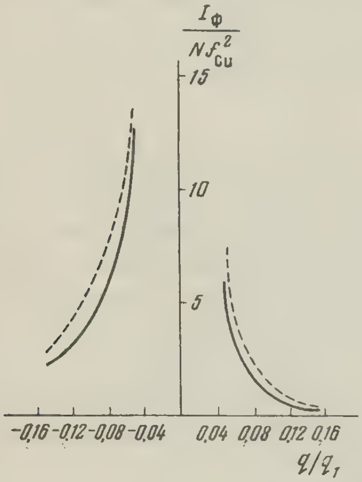
*For crystals of finite size in the region of very small q ($qa \sim 1$, where a are the dimensions of the crystal) the formulas obtained above cease to hold at the critical point. The region indicated above in the space of the reciprocal lattice is occupied by extended Laue reflection as a result of diffraction by a crystal of finite size.

†It should be noted that in the neighborhood of the critical point it is necessary to satisfy in place of the criterion (9) a more strict criterion, which is obtained from (9) by replacing k_m by $\sqrt{\varphi_{cc}/\beta}$.



For the same solution Cu_3Au we have also evaluated the factor 2B with the aid of formula (28). In doing this we have assumed $\sigma = 0.36$ (calculated on the assumption that σ depends linearly on c). The calculated value of 2B is equal to $0.43\,A^{-2}$, while the measured value is equal to $0.42\,A^{-2}$ according to the data of Borie,[9] and is equal to $0.34\,A^{-2}$ according to the data of Herbstein, Borie and Auerbach.[3] The agreement of the measured values of 2B with those calculated with the aid of the formula for ideal solutions obtained in the approximation of an isotropic continuum is apparently related to the smaller influence of short range order on L than on I_ϕ, and also to a partial compensation of errors introduced by the different approximations.

2. ORDERED SOLUTIONS

It is not difficult to generalize the results obtained above to the case of ordered solutions of arbitrary structure having a center of symmetry. The distribution of atoms A and B among the lattice points of the solution may, as before, be characterized by the numbers c_s or by their Fourier components $c_{\mathbf{k}}$. However, in ordered solutions the lattice is subdivided into lattice points of several kinds and the average atomic concentrations $\bar{c}_\gamma$ of atoms A on the lattice points of different kinds γ become different. The coefficients w and V in the expansion (18) for the free energy now depend not only on the distance between the lattice points s and s′, but also on the kind of lattice points. Therefore the wave of displacements of the lattice corresponding to the **k**-th fluctuation wave has different amplitudes $a_{\mathbf{k}\gamma}$ [see formula (2)] for lattice points of different kinds. These amplitudes are determined from a system of linear equations which are obtained by substituting values corresponding to the **k**-th wave into expressions of the type (19) for lattice points of different sorts, in place of $c_s - c_\gamma$ and $\delta R_{s'j'}$, and by equating F_{si} to zero:

$$iV_{ss'ij}^{\gamma\gamma'}n'_{j\gamma'}\frac{a_{\mathbf{k}\gamma'}}{k}\exp(i\mathbf{k},\ \mathbf{R}_s - \mathbf{R}_{s'}) = w_{ss'i}^{\gamma\gamma'}\exp(i\mathbf{k},\ \mathbf{R}_s - \mathbf{R}_{s'}) \tag{29}$$

In the limit of long wavelengths, it may be easily seen that $a_{\mathbf{k}\gamma}$ for different γ are the same and, as before, are determined by formula (16) of the theory of elasticity.

In decomposing the atomic factors into their average values and deviations from the average in an ordered crystal, one must introduce the average atomic factors $\bar{f}_\gamma = \bar{c}_\gamma f_A + (1 - \bar{c}_\gamma) f_B$ for lattice points of each kind. The averaged structure obtained in this way possesses lower symmetry than the solution with a disordered distribution of atoms among the lattice points of the same lattice, as a result of which each elementary cell of the reciprocal lattice of the above disordered solution will contain lattice points corresponding to superstructural reflections, in addition to the lattice points corresponding to structural reflections (which remain when disordering takes place). With the aid of formulas (3) and (2), the expression for the intensity of scattering by an ordered solution in the approximation adopted above may be written in the form:*

$$I = 8\pi^3 \frac{N_0}{\Delta'} \sum_l |f_l|^2 \delta(\mathbf{q}_l) + N_0^2 \overline{\left| \sum_{l=0}^{\nu-1} c_{\mathbf{q}l} f_l' \frac{\mathbf{q}_1 \mathbf{q}_l'}{q_l^2} a_{\mathbf{q}l1} - \nu (f_A - f_B) c_{\mathbf{q}0} \right|^2}. \qquad (30)$$

Here N_0 is the number of elementary cells of the ordered solution; Δ' is the volume of these cells; the index ℓ denotes lattice points in any cell of the reciprocal lattice of the above solution which are characterized by the vectors $\mathbf{K}_\ell$ (with the value $\ell = 0$ corresponding to a structural reflection); $\mathbf{q}_\ell = \mathbf{q}_1 - 2\pi \mathbf{K}_\ell$ is the vector $\mathbf{q}_1$ "reduced" to one of the elementary cells of the reciprocal lattice of the ordered crystal lying in the central elementary cell of the disordered crystal; $c_{\mathbf{q}\ell}$ is the Fourier coefficient of the quantities c_s corresponding to the vector $\mathbf{q}_\ell$; $a_{\mathbf{q}\ell_1}$ is the coefficient $a_{\mathbf{q}\gamma}$ for lattice points of the first kind $(\gamma = 1)$ corresponding to this vector;

$$f_l = \sum_{\gamma=1}^{\nu} \bar{f}_\gamma \exp(2\pi i \mathbf{K}_l \mathbf{h}_\gamma) \exp(-L_\gamma/2)$$

is the structure factor for the ℓ-th reflection; ν is the number of lattice points in a cell; $\mathbf{h}_\gamma$ is a vector drawn from the central lattice point of the cell to the lattice point of index γ;

$$\mathbf{q}_l' f_l' = \sum_{\gamma=1}^{\nu} \bar{f}_\gamma (a_{\mathbf{q}l\gamma}/a_{\mathbf{q}l1}) \exp(-L_\gamma/2) \cdot \exp(2\pi i \mathbf{K}_l \mathbf{h}_\gamma) q_l \mathbf{n}'_\gamma$$

$(f_\ell' \rightarrow f_\ell$ as $\mathbf{q} \rightarrow 0)$, and $\exp(-L_\gamma/2)$ is a factor which takes into account the reduction in the scattering amplitude by atoms situated at lattice points of the γ kind, owing to geometrical distortions. L_γ are different for lattice points of different kind. In the interest of brevity we shall not give here formulas for these quantities. (They may be determined experimentally from the reduction in the intensity of lines on an x-ray photograph by means of the usual methods of analyzing such x-ray photographs.) We note that $L_\gamma \sim (\sin\theta/\lambda)^2$ and that for non-cubic crystals they are anisotropic.

In calculating the average value of (30) it should be kept in mind that, in contrast to the case of disordered solutions, nonzero values are obtained not only for mean squares but also for the mean values of the products of the Fourier components $c_{\mathbf{q}}$ with different $\mathbf{q}_\ell$ and $\mathbf{q}_j$ which, however, correspond to the same value of the vector $\mathbf{q}_1$. These mean values can be easily expressed in terms of the correlation parameters, starting with formula (1) for $c_{\mathbf{k}}$ (in which in the terms involving lattice points of each kind one should replace c by the corresponding value of c_γ):

$$\overline{|c_{\mathbf{q}l}|^2} = \frac{1}{N\nu} \sum_{\gamma=1}^{\nu} [\bar{c}_\gamma (1 - \bar{c}_\gamma) - \sum_{\rho \neq 0} \varepsilon_\gamma(\rho) \cos \mathbf{q}_l \rho],$$

$$\frac{1}{2} \overline{(c_{\mathbf{q}l} c_{\mathbf{q}j}^* + c_{\mathbf{q}l}^* c_{\mathbf{q}j})} = \frac{1}{N\nu} \sum_{\gamma=1}^{\nu} [\bar{c}_\gamma (1 - \bar{c}_\gamma) \cos(2\pi \mathbf{h}_\gamma, \mathbf{K}_l - \mathbf{K}_j) - \sum_{\rho \neq 0} \varepsilon_\gamma(\rho) \cos(\mathbf{q}_l \rho + 2\pi \mathbf{h}_\gamma, \mathbf{K}_l - \mathbf{K}_j)]. \qquad (31)$$

Here $\mathbf{q}_\ell - \mathbf{q}_j = 2\pi(\mathbf{K}_j - \mathbf{K}_\ell)$, while the correlation parameters are given by

$$\varepsilon_\gamma(\rho) = [\bar{c}_\gamma \bar{c}_{\gamma'} - \overline{c_s c_{s'}^{(\gamma\rho)}}],$$

where the averaging is carried out over pairs of lattice points separated from each other by the distance ρ, while s is a lattice point of the γ kind. Substituting (31) into (30) we obtain

$$I = 8\pi^3 \sum_l |f_l|^2 \delta(\mathbf{q}_l) + \frac{N_0}{\nu^2} \Big\{ \sum_{l=0}^{\nu-1} \sum_{\gamma=1}^{\nu} [\bar{c}_\gamma (1 - \bar{c}_\gamma) - \sum_{\rho \neq 0} \varepsilon_\gamma(\rho) \cos \mathbf{q}_l \rho] [a_{\mathbf{q}l1} f_l' \mathbf{q}_1 \mathbf{q}_l' q_l^{-2} - \nu (f_A - f_B) \delta_{l0}]^2$$
$$+ \sum_{l,j=1}^{\nu-1} \sum_{\gamma=1}^{\nu} [\bar{c}_\gamma (1 - \bar{c}_\gamma) \cos(2\pi \mathbf{h}_\gamma, \mathbf{K}_l - \mathbf{K}_j) - \sum_{\rho \neq 0} \varepsilon_\gamma(\rho) \cos(\mathbf{q}_l \rho + 2\pi \mathbf{h}_\gamma, \mathbf{K}_l - \mathbf{K}_j)]$$
$$\times [a_{\mathbf{q}l1} f_l' \mathbf{q}_1 \mathbf{q}_l' q_l^{-2} - \nu (f_A - f_B) \delta_{l0}] \times [a_{\mathbf{q}j1} f_j' \mathbf{q}_1 \mathbf{q}_j q_j^{-2} - \nu (f_A - f_B) \delta_{j0}]. \qquad (32)$$

Thus, just as in the case of absence of distortions, the scattering intensity is expressed in terms of the concentrations on lattice points of different kinds and in terms of the correlation parameters. Formula (32) together with equations (29) enables us to determine I_ϕ from given values of $\bar{c}_\gamma$ and $\epsilon_\gamma(\rho)$ and also enables us, in a number of cases, to determine $\epsilon_\gamma(\rho)$ (or certain combinations of them) from an experimentally determined intensity distribution I_ϕ. From (32) it can be seen that in the neighborhood of both the structural and the superstructural reflections I_ϕ increases proportionally to q^{-2}. The principal terms proportional to q^{-2} may be represented in the neighborhood of the lattice points of the reciprocal lattice in a simple form if one makes use of formula (26) for $|\overline{c_{\mathbf{q}_l}}|^2$:

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*In the discussion of ordered solutions it is assumed that there are no anti-phase domains in the crystal. The existence of such non-equilibrium domains, which arise if the relaxation times for ordering are large, leads to a smearing of the intensity distribution.

$$I_\phi \approx (N_0/\Delta')\, a_{\mathbf{q}l}^2 (\mathbf{q}_1\mathbf{q}'_l)^2\, q_l^{-4}\, |f_l|^2\, kT\, (\varphi_{cc}+\beta q^2)^{-1}. \quad (33)$$

As may be seen from (33), the intensity of this part of the diffuse scattering, just like the intensity of the corresponding regular reflection, is proportional to the square of the modulus of the structure factor f_l. If $|a_{\mathbf{q}l}\bar{f}_l| \ll |f_A - f_B|$, then in the neighborhood of structural reflections one may, as before, use formula (27) for the determination of I_ϕ (and not only of its part proportional to q^{-2}), i.e , one may determine I_ϕ with the aid of independent thermodynamic measurements.

For a more detailed investigation of the dependence of I_ϕ on the temperature in the vicinity of regular reflections it is convenient to introduce parameters of long-range order. Such a discussion will be carried out for the sake of simplicity for ordered solutions with two lattice points in an elementary cell (for example solutions of the type of β-brass or NaCl), in which one may restrict oneself to specifying a single long-range order parameter η. In this case one may characterize the distribution of atoms A and B among the lattice points of the lattice by specifying the numbers c'_t and η_t, where t is the cell index, c'_t is equal to the arithmetic mean of the numbers c_s for the two lattice points of the cell, and η_t is equal to the difference of these numbers. The average values of the quantities c'_t and η_t coincide with c and η (which are related to $\bar{c}_\gamma$ by the expressions $\bar{c}_1 = c + \eta/2$, $\bar{c}_2 = c - \eta/2$). In place of c'_t and η_t one may introduce their Fourier components $c'_{\mathbf{k}}$ and $\eta_{\mathbf{k}}$ for $\mathbf{k}/2\pi$ lying in the first cell of the reciprocal lattice of the ordered crystal. Then instead of (2) the $\delta\mathbf{R}_s$ are determined by the formula

$$\delta\mathbf{R}_s = \sum_{\mathbf{k}} p_\gamma a_{\mathbf{k}} \mathbf{k}' k^{-2} i\, [c'_{\mathbf{k}} \exp(-i\mathbf{k}\mathbf{R}_s) - c'^{*}_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{R}_s)]$$

$$+ \sum_{\mathbf{k}} p'_\gamma b_{\mathbf{k}} \mathbf{k}' k^{-2} i\, [\eta_{\mathbf{k}} \exp(-i\mathbf{k}\mathbf{R}_s) - \eta^{*}_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{R}_s)]. \quad (34)$$

Here p_γ and p'_γ reduce to unity as $\mathbf{k} \to 0$ and $a_{\mathbf{k}}$, as in the case of disordered solutions, is determined by formulas (16), (17), while $b_{\mathbf{k}}$ is determined by analogous formulas in which c is replaced by η.

Evaluating the average value of (4) by means of (34) and of the Gaussian probability distribution for $c'_{\mathbf{k}}$ and $\eta_{\mathbf{k}}$ [formula (9) of I], in the same way in which this was done for disordered solutions, we find that, in the neighborhood of structural reflections, I_ϕ is equal to

$$I_\phi = \frac{N}{\Delta}\, \frac{kT}{(\varphi'_{\eta\eta}+\alpha q^2)(\varphi'_{cc}+\beta' q^2) - (\varphi'_{\eta c}+\gamma q^2)^2} \Big\{ (\varphi'_{\eta\eta}+\alpha q^2) \Big[\frac{a_{\mathbf{q}} f\, (\mathbf{q}_1\mathbf{q}')}{q^2} - f_A + f_B \Big]^2 - 2(\varphi'_{\eta c}+\gamma q^2)\, b_{\mathbf{q}} \frac{(\mathbf{q}_1\mathbf{q}')}{q^2} \bar{f} \Big[\frac{a_q \bar{f}\, (\mathbf{q}_1\mathbf{q})}{q^2} - f_A + f_B \Big]$$

$$+ (\varphi'_{cc}+\beta' q^2)\, b_{\mathbf{q}}^2 \frac{(\mathbf{q}_1\mathbf{q}')^2}{q^4} \bar{f}^2 \Big\} \approx \frac{N}{\Delta}\, \frac{kT}{(\varphi'_{cc}+\beta' q^2) - (\varphi'_{\eta c}+\gamma q^2)^2 / (\varphi'_{\eta\eta}+\alpha q^2)^2} \Big[\frac{a_{\mathbf{q}} \bar{f} (\mathbf{q}_1\mathbf{q}')}{q^2} - f_A + f_B \Big]^2 \quad (35)$$

In the neighborhood of superstructural reflections I_ϕ is given by the formula

$$I_\phi = \frac{N}{4\Delta}\, \frac{(f_A - f_B)^2\, kT}{(\varphi'_{\eta\eta}+\alpha q^2)(\varphi'_{cc}+\beta' q^2) - (\varphi'_{\eta c}+\gamma q^2)^2} \Big[(\varphi'_{cc}+\beta' q^2) \Big(1 - \eta b_{\mathbf{q}} \frac{\mathbf{q}_1\mathbf{q}'}{q^2}\Big)^2 - 2(\varphi_{\eta c}+\gamma q^2)\, \eta a_{\mathbf{q}} \frac{\mathbf{q}_1\mathbf{q}'}{q^2} \Big(\eta b_{\mathbf{q}} \frac{\mathbf{q}_1\mathbf{q}'}{q^2} - 1 \Big)$$

$$+ (\varphi_{\eta\eta}+\alpha q^2)\, a_{\mathbf{q}}^2 \eta^2 \frac{(\mathbf{q}_1\mathbf{q}')^2}{q^4} \Big] \approx \frac{N}{4\Delta}\, \frac{(f_A - f_B)^2\, kT}{(\varphi'_{\eta\eta}+\alpha q^2)(\varphi'_{cc}+\beta' q^2) - (\varphi'_{\eta c}+\gamma q^2)^2}$$

$$\times \Big[\varphi'_{cc} + \beta' q^2 + 2(\varphi'_{\eta c}+\gamma q^2)\, \eta a_{\mathbf{q}} \frac{\mathbf{q}_1\mathbf{q}'}{q^2} + (\varphi'_{\eta\eta}+\alpha q^2)\, a_{\mathbf{q}}^2 \eta^2 \frac{(\mathbf{q}_1\mathbf{q}')^2}{q^4} \Big]. \quad (36)$$

In the above, $\mathbf{q}/2\pi$ characterizes in both cases the deviation of a point in the space of the reciprocal lattice from the corresponding lattice point. In the expressions, which appear in (35) and (36) after the sign of approximate equality, terms containing $b_{\mathbf{q}}$ have been neglected. For small q these terms are proportional to ϵ^2, where ϵ is a parameter which characterizes the difference in the atomic radii of atoms A and B in the solution (for small $\eta b_{\mathbf{q}}$ they are also proportional to η), while $a_{\mathbf{q}} \sim \epsilon$. Therefore the retention of terms containing $b_{\mathbf{q}}$ would not be quite consistent within the framework of the approximation under discussion, when fluctuations in the short-range order parameters, which also lead to effects proportional to ϵ^2, are not taken into account (the short range order itself is, of course, taken into account). The possibility of neglecting the above terms is justified by the observation that for most solutions (for example AuCu) v changes considerably less as a result of ordering than of going over from component A to B. However, cases are possible when taking into account terms containing $b_{\mathbf{q}}$ would improve the expression for I_ϕ.

For sufficiently small q one may neglect terms on the order of q^2 compared to φ'_{cc}, $\varphi'_{\eta c}$ and

$\varphi'_{\eta\eta}$. In this case the following expression appears in the denominator of formula (35)

$$\varphi'_{cc} - \varphi'^{2}_{\eta c} / \varphi'_{\eta\eta} = \varphi_{cc} = \frac{kT}{(1-c)\Delta} \frac{d \ln a_A}{dc},$$

where in the evaluation of the derivative d/dc (in contrast to ∂/∂c) the dependence of η on c is taken into account. At the temperature T_0 of the phase transition of the second kind into the ordered state, I_ϕ varies sharply in the vicinity of structural reflections, but remains finite. The dependence of I_ϕ on **q** in the vicinity of superstructural reflections close to the temperature T_0 differs essentially from the corresponding dependence near structural reflections. While near structural reflections I_ϕ always contains a term proportional to q^{-2}, near points of the reciprocal lattice corresponding to superstructural reflections for $T > T_0$ $(\eta = 0)$ such a term is missing entirely, while for $T < T_0$ it is proportional to $\eta^4 (\varphi'_{\eta\eta}/\varphi'_{cc} \sim \eta^2, \varphi'_{\eta c}/\varphi'_{cc} \sim \eta)$ and begins to play an essential role only for very small q, when $q/q_1 \sim a_{\mathbf{q}}\eta^2$. For $T > T_0$, the expression for I_ϕ turns out to be the same as in the absence of geometrical distortions [see formula (10) in I]. Since at $T = T_0$ $\varphi'_{\eta\eta} = 0$ and $\varphi'_{\eta c} = 0$, in the neighborhood of this temperature the first term in (36) determines the anomalously large scattering in the neighborhood of superstructural reflection which was discussed in Ref. 18 for single component crystals and in I for solid solutions.

At the critical point at which the curve of phase transitions of the second kind goes over into the decay curve (point 0 in Fig. 1 of I) the condition[19] $\varphi'_{cc}\varphi'_{\eta\eta} = \varphi'^{2}_{\eta c}$ is fulfilled in the ordered phase. Therefore, as is evident from (35), in the neighborhood of this point, I_ϕ in the ordered phase becomes especially large not only near superstructural reflections, but also near structural reflections where the terms $\sim q^{-2}$ are also anomalously large. In the disordered phase, $\varphi'_{\eta c} \equiv 0$ and I_ϕ must be considerably smaller in the vicinity of structural reflections.

The expression for I_ϕ assumes a simple form also in the case of almost completely ordered solutions $(T \ll T_0)$. In this case, as shown by the author,[20] one may neglect $\epsilon_\gamma(\rho)$ in comparison with the products $\bar{c}_\gamma(1-\bar{c}_\gamma)$ (also small). Therefore in the general formula (32) one may neglect terms containing $\epsilon_\gamma(\rho)$, so that I_ϕ turns out to be expressed only in terms of the parameters $\bar{c}_\gamma$, which characterize the long-range order. In the special case of structures considered above, which are characterized by a single long range order parameter, we shall obtain from (32)

$$I_\Phi = N\,[c\,(1-c) - \eta^2/4]\,\{[a_{\mathbf{q}}\bar{f}\,(\mathbf{q}_1\mathbf{q}')/q^2 - f_A + f_B]^2 + (f_A - f_B)^2\, a^2_{\mathbf{q}'}\,(\mathbf{q}_1\mathbf{q}'_s)^2/q'^4\}, \tag{37}$$

where $\mathbf{q}/2\pi$ and $\mathbf{q}_s/2\pi$ characterize distances to lattice points corresponding to structural and superstructural reflections. For $\eta \approx 1$, as was shown by the author,[20] $c(1-c) - \eta^2/4$, and consequently also I_ϕ, decrease exponentially as the temperature is lowered.

The formulas obtained above have been derived for binary substitution solutions. They may also be applied to the determination of I_ϕ in the vicinity of regular reflections by occlusion solutions in which atoms are occluded between the lattice points of the crystalline lattice of the pure metal. By means of the method developed above it is not difficult to obtain formulas for I_ϕ in such solutions, valid for arbitrary **q**. We note that with the aid of formula (21) it is also possible to determine the intensity of scattering by vacancies (or by complexes of vacancies) if c is interpreted as their concentration (equilibrium or non-equilibrium). The method developed above may also be used for the investigation of scattering by more complicated defects, for example by precipitations formed on aging or by dislocations.

In conclusion we note that the method used here, in which the geometrical distortions are related to fluctuations of composition or of internal parameters, may also be employed for the solution of other problems related to geometrical distortions in solution, for example for the determination of its electrical resistance, of the elastic energy of the distortions, of the mean squared displacements of the atoms, etc.

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Letters to the Editor

EXCITATION OF OPTICAL FREQUENCIES IN CRYSTALS BY SLOW NEUTRONS

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THE inelastic single-phonon scattering of slow neutrons with the excitation of an acoustical oscillation has been studied both from the theoretical and experimental points of view. In crystals with many atoms, neutron scattering can excite not only acoustical, but also optical modes of vibration in the lattice. Experimental difficulties make the identification of different modes of the dispersion function $\omega(\mathbf{q})$ either very difficult or impossible. However, it is possible to show that if the speed v of the incident neutrons is greater than the maximal speed of sound $c_{\max}$, and if they are incident at a Bragg angle, then the reflected beam will contain, in addition to elastically scattered neutrons, only neutrons scattered in the optical mode (not on the acoustical one). This circumstance can be used to determine optical frequencies experimentally.

The unavoidable presence of an angular spread Δ in the incident beam will lead to energies $E_0 - h\omega_{ac}$ in the spectrum of the scattered neutrons, which correspond to the excitation of acoustical frequencies (E_0 is the initial neutron energy). Let us see what limitations must be placed on Δ if we wish to separate the optical from the acoustical frequencies. It is clear that the maximal wave vector $q_{\max}$ of the emitted or absorbed phonon will be $\approx k_0 v$, where $k_0 = mv/h$ is the wave vector of the neutron. Taking v to be as small as it can, $c_{\max}$, we obtain

$$(h\omega_{ac})_{\max} = mc_{\max}^2\Delta.$$

and hence obtain the desired relations:

$$\Delta \ll h\omega_{\text{opt}}/mc_{\max}^2.$$

Now let us calculate the cross section for scattering with excitation of an optical mode. We denote the wave vector of the scattered neutron by $\mathbf{k}$, and that of the emitted phonon by $\mathbf{q}$. Generalizing formula (40.4) of Ref. 1 to a complex crystal, we obtain the following formula for the cross section of a given process, calculated per unit cell:

$$\frac{d\sigma}{d\mathbf{k}\,d\mathbf{q}} = \frac{1}{k_0}\left|\sum_\alpha m_\alpha^{-1/2}a_\alpha e^{-W_\alpha}\exp\{i(\boldsymbol{\varkappa}-\mathbf{q})\mathbf{r}_\alpha\}\,\boldsymbol{\varkappa}\mathbf{e}_{\mathbf{q}t}^{\alpha *}\right|^2$$

$$\frac{h(\bar{n}_{\mathbf{q}t}+1)}{\omega_{\mathbf{q}t}}\,\delta(\boldsymbol{\varkappa}-\mathbf{q}-2\pi\boldsymbol{\tau})\,\delta\left(k^2-k_0^2-\frac{2m}{h}\,\omega_{\mathbf{q}t}\right). \quad (1)$$

Here $\boldsymbol{\kappa} \equiv \mathbf{k}_0 - \mathbf{k}$; the index α labels the nuclei in unit cell; $\mathbf{r}_\alpha$, m_α, a_α, e^{-W_α} are the corresponding coordinates, masses, scattering amplitudes, and thermal factors; $\omega_{\mathbf{q}t}$ and $\bar{n}_{\mathbf{q}t}$ are the frequencies and mean occupation numbers of the oscillation $(\mathbf{q}, t)$; t labels the modes; $\mathbf{e}_{\mathbf{q}t}^\alpha$ are the displacement vectors of nucleus α in the oscillation $(\mathbf{q}, t)$, normalized as follows:

$$\sum_\alpha \mathbf{e}_{\mathbf{q}t}^\alpha \mathbf{e}_{\mathbf{q}t'}^{\alpha *} = \delta_{tt'}. \quad (2)$$

$\omega_{\mathbf{q}t}$ depends little on $\mathbf{q}$ for the optical modes. For example, according to the calculations of Tolpigo and Zaslavs'ka,[2] $\omega_{\mathbf{q}t}$ varies in NaCl by no more than 20% over the whole range of $\mathbf{q}$. Since, for our case, only small $\mathbf{q}$ appear.

$$\omega_{\mathbf{q}t} \approx \omega_{0t}, \qquad \bar{n}_{\mathbf{q}t} \approx \bar{n}_{0t} = (e^{h\omega_{0t}/T}-1)^{-1}.$$

Then from (1) it follows that

$$\frac{d\sigma}{d\Omega} = \sqrt{1-\frac{2m\omega_{0t}}{hk_0^2}}\left|\sum_\alpha m_\alpha^{-1/2}a_\alpha e^{-W_\alpha}e^{i\varkappa\mathbf{r}_\alpha}\boldsymbol{\varkappa}\mathbf{e}_{\mathbf{q}t}^{\alpha *}\right|^2$$

$$\times\frac{h}{2\omega_{0t}}(1-e^{-h\omega_{0t}/T})^{-1}. \quad (3)$$

As an example, let us consider the excitation of longitudinal optical vibrations in a crystal of NaCl. Since these are polarized along the main diagonal of the cubical NaCl lattice, we should take a Bragg angle such that $\boldsymbol{\kappa}$ is parallel to the main diagonal, i.e., $\boldsymbol{\kappa} = 2\pi p(\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2 + \boldsymbol{\tau}_3)$, where the $\boldsymbol{\tau}_i$ are the basis vectors of the inverse lattice, and p is an integer. We find the vectors $\mathbf{e}_{\mathbf{q}\,\text{long}}$ for the nuclei Na and Cl in the limit $q \to 0$. Since, for $q = 0$, the set of ions Na and Cl vibrate relative to each other bodily, these vectors can be found from the center-of-mass condition and the normalization (2). The thermal factors are calculated in the Einstein approximation. The optical frequencies are obtained from Ref. 2. The neutron velocity v appears only in the first factor in (3), which is always close to 1 for $v > c_{\max}$. We can neglect it. The result is the following differential scattering cross section for one molecule NaCl (in millibarns per steradian):

p	1	2	3	4	5	6
dσ/dΩ	7.1	25	44.6	59	62.5	54.5

As a function of p, dσ/dΩ has a maximum due to competition between the factor κ^2 and the thermal factor.

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EQUATION OF STATE FOR THE MgO CRYSTAL

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THIS letter is a report on the calculation of the energy of the MgO crystal by a statistical method. The dependence of energy on volume at absolute zero has been found for pressures from zero to 10^7 atmospheres (at high pressures, the effect of temperature is small[1]).

In calculations for lattices of the alkali halide type it is customary to use a potential derived from the electron density of an isolated ion, i.e., to ignore the deformation of the ions in the lattice. But an isolated oxygen ion is not, in general, stable; in this work the electron density chosen for oxygen was found for a lattice at zero pressure by a variational method.[2] The magnesium ion deforms very litte, and the Hartree-Fock electron density is used in this case.[3] The energy calculations took into account the first quantum correction,[4] assuming an overlap of about 20%.

According to the statistical theory, the energy of any system of electrons with a density ρ is equal to:

$$\varepsilon(\rho) = \int\left\{\varkappa_k\rho^{5/3} + \left(V_k + \frac{1}{2}V_p\right)\rho - \varkappa_a\rho^{4/3} - K(\rho) + \frac{1}{72}\frac{(\nabla\rho)^2}{\rho}\right\}dv,$$

$$\varkappa_k = 2,871, \quad \varkappa_a = 0.7386,$$

i.e., the sum of the kinetic (Fermi) and potential (Thomas-Fermi) energies of the electrons in the nuclear field, the Dirac exchange energy, the Wigner-Macke correlation energy, and the quantum correction of Kompaneev and Kirzhnits.[4-6] Atomic units are used throughout.

The correlation energy,

$$K(\rho) = 0.0306\rho\left\{\ln\left(1 + \frac{4}{\lambda}\right) - \frac{2}{1 + 2\lambda}\right\}, \quad 1/\lambda = 4.862\rho^{1/3}$$

in the region of density $\rho = 0.2$ to 0.02 which we consider here, is approximately equal to

$$K(\rho) \approx 0.113\varkappa_a\rho^{4/3}.$$

The ionic energy of the crystal is obtained by summing the interaction energy of one pair of ions over all the ion pairs in the lattice. The interaction energy of each pair of ions is equal to

$$U = \varepsilon(\rho_1 + \rho_2) - \varepsilon(\rho_1) - \varepsilon(\rho_2) = q_1 q_2 / r + U_s,$$

where q_1, q_2 are the charges on the ions, ρ_1, ρ_2 are the electron densities of the ions, and U_s is the energy required to remove the electron clouds from both ions.[5] The total energy of the crystal, referred to a single ion pair, has the form

$$E(\delta) = -\alpha q^2/\delta + 6U_s^{+-}(\delta) + 6U_s^{--}(\sqrt{2}\delta) + 6U_s^{++}(\sqrt{2}\delta),$$

where $\alpha = 1.7476$ for a face-centered lattice, δ is the shortest distance between unlike ions, and U_s^{+-}, U_s^{--}, U_s^{++} are the overlap energies of the corresponding neighboring like and unlike ions.

In order to facilitate the numerical calculations, the electron densities of the ions taken from the literature[2,3] were approximated by the formulas

$$\rho_1(r) = 45\exp(-4.9r) \quad \text{magnesium.}$$

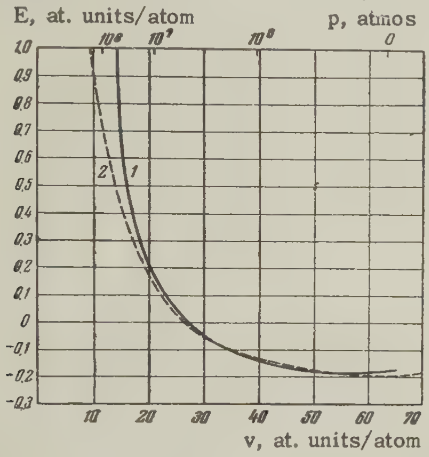
$$\rho_2(r) = \left\{\begin{array}{lll} 280\exp(-15r) & \text{for} & r < 0.35 \\ 3.85\exp(-2.3r) & \text{"} & 0.35 < r < 1.6 \\ 1.54\exp(-1.8r) & \text{"} & r > 1.6 \end{array}\right\} \text{oxygen.}$$

Curves of crystal energy, referred to the energies of the free atoms, are shown in the figure (in atomic units). Curve 1 is the theoretical one. The lower part of Curve 2, where $p < 3 \times 10^4$ atmos, is based on Bridgman's experimental data[7] extrapolated to absolute zero; the portion of Curve 2 where $p > 3 \times 10^4$ atmos is an extrapolation of Bridgman's data to high pressures by Davydov's formula.[1]

In the range of pressures from zero to 10^6 atmospheres the theoretical Curve 1 can be approximated by the formula

$$E(v) = 93.7\exp(-1.69v^{1/3}) - \\ -3.495v^{-1/3}+0.582 \text{ at. units/atom.}$$

The pressure is found from the formula $p = -dE/dv$. The equilibrium volume at zero pressure is calculated to be $v_0 = 55$ at. units/atom, and is found experimentally to be $v_0 = 63$. The calculated binding energy, referred to the energy of the free atoms, is 10.0 ev per molecule = 231 kcal/mole. The experimental value of this energy[8] is 242 kcal/mole. At pressures above 5×10^6 atmos the calculated curve 1 lies somewhat higher than curve 2. This might be explained by the onset of an energy storage due to the overlapping of the inner shells. In curve 2 it is only the overlapping of the outer shells which is extrapolated, whereas the calculations for curve 1 automatically take into account the overlapping of the inner shells as well.



I wish to express my thanks to B. I. Davydov for proposing this subject and discussing the results.

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31

EFFECT OF ELECTRIC POLARIZATION ON MAGNETIC PROPERTIES OF FERRITES

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If the spin-orbit interaction of the d electrons in ferrites is taken into account, the effective magnetic permeability becomes a function of the electric polarization of the electron shells of the atoms, i.e., of the dielectric constant. No such dependence is observed at radio frequencies in ferromagnetic metals, owing to the strong polarizability of the conduction electrons.

In certain types of ferrites one can expect the electric properties to affect not only the frequency dependence of the effective permeability, or the width, form, and position of the resonance-absorption line, but also the dependences of these effects on the dimensions and shapes of the specimens, dependences that cannot be accounted for by the method of demagnetizing factors.

The spin-orbit interactions, within the framework of the macroscopic theory, can be accounted for with the aid of the following closed system of equations:

$$\dot{\mathbf{M}} = \gamma \mathbf{M} \times \mathbf{H}_{\text{eff}}; \tag{1}$$

$$\ddot{\mathbf{P}}_j + \alpha_j \dot{\mathbf{P}}_j + \omega_j^2 \mathbf{P}_j = -(g/c)\,\partial \mathbf{A}/\partial t, \tag{2}$$

$$\Delta \mathbf{A} = -\frac{4\pi}{c}\left[c \operatorname{curl} \mathbf{M} + \dot{\mathbf{P}} - \frac{1}{4\pi c}\ddot{\mathbf{A}}\right], \tag{3}$$

$$\mathbf{P} = \sum_j \mathbf{P}_j, \quad \mathbf{H}_i = -4\pi\mathbf{M} + \operatorname{curl}\mathbf{A};$$

$$\mathbf{H}_{\text{eff}} = \mathbf{H} + aM_s^{-2}\Delta M + \mathbf{H}_i,$$

H is the intensity of the external field, and $aM_S^{-2} \times \Delta\mathbf{M}$ are the fields of the exchange forces. The self-consistent internal field $\mathbf{H}_i$ is due to the spin-spin (source — c curl **M**) and spin-orbit (source — $\dot{\mathbf{P}}$) interactions.

The term **P** in Eq. (2) describes the polarization current due to the change in the electric dipole moment of the atom, a change caused by the vortical electric field due to the magnetization fluctuations. In other words, it is due to the changes in the spatial portion of the wave function of the d electrons, occurring upon change of orientation of the magnetic moments of the electrons.

Equations (2) describe the change in the polarization of a set of interacting atoms with an internal electromagnetic field.

In the work of Ament and Rado,[1] the effective magnetic permeability μ_{eff} is calculated for a ferromagnetic metal having an electric conductivity, and its influence on the width, shape, and position of the resonance line is estimated. For ferromagnetic conductors placed in a radio-frequency field, the displacement and polarization currents can be neglected, as was done in Ref. 1. For ferrites, this cannot be done in general, and therefore Eq. (3) takes these currents into account.

Using the original system of equations to solve the boundary problem for a half-space filled with ferrite, in the case of a plane wave normally incident on the boundary, we obtain, in analogy with the procedure given in Ref. 1, the impedance $Z = [\mu_{\text{eff}}/\epsilon_{\text{eff}}]$, where according to (2) and (3)

$$\varepsilon_{\text{eff}} = \varepsilon' - i\varepsilon'' = 1 + 4\pi \sum_i g_i (\omega_j^2 - \omega^2 + i\omega\alpha_j)^{-1}. \tag{4}$$

Then

$$\mu_{\text{eff}} = \frac{\eta - \Omega^2 + 2i\,(a\omega^2/2\pi M_s^2 c^2)^{1/2} (\varepsilon' - i\varepsilon'')^{1/2}}{[\eta - \Omega^2 + i\,(a\omega^2/2\pi M_s^2 c^2)^{1/2} (\varepsilon' - i\varepsilon'')^{1/2}]^2}. \tag{5}$$

Here η and Ω stand for the same quantities as in Ref. 1, and the quantity a is denoted in Ref. 1 by A.

It follows from (5) that the electric polarization of the ferrite is accounted for by a shift of the resonant frequency and to a broadening of the resonance-absorption line.

The results by Ament and Rado are contained in Eq. (5). In fact, the conduction current can be taken into account by putting $j = 0$ in (4), for the case of the free electrons. Then $\omega_0 = 0$, α_0 is the frequency of collision between the electrons and the lattice, and $4\pi g_0$ is the square of the Langmuir frequency. Now $\sigma = g_0/\alpha_0$ is the static value of the electron conductivity. In metals $\sigma/\omega \gg 1$, and $\omega \ll \alpha_0$, and therefore (5) changes into (31b) of Ref. 1.

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32

CROSS SECTIONS FOR INTERACTION OF PI MESONS WITH CARBON NUCLEI

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IN the optical model of the nucleus, scattering and absorption of particles is described as the motion of these particles in the field of a complex potential. A considerable part of the work done on the optical model is devoted to a determination of the complex potential or of uniquely-related parameters which provide best agreement with experimental data on the interaction of fast particles with nuclei. In a number of other analyses[1,2] it was sought to relate these parameters, which characterize the interaction of fast particles with nuclei, to the scattering cross sections and amplitudes for elementary particles. In particular, Frank, Gammel, and Watson[1] have obtained formulas relating the complex potential of the optical model to the amplitude of the forward scattering of mesons against nucleons, averaged over the protons and neutrons of a nucleus,* and have calculated the values of the real and imaginary parts of the potential, and thus of the mean free path λ_t of mesons in nuclear matter at energies of 0 to 350 Mev. This data will be used in the present article to calculate the total effective cross sections for elastic and inelastic interactions of π mesons with carbon nuclei in this energy region. The computations are carried out according to the formulas of the quasi-classical approximation.[4,5]

$$\sigma_{\text{el}} = \pi\lambda\!\!\!^{-}{}^2 \sum_{l=0}^{l\leqslant R/\lambda\!\!\!^{-}} (2l+1)\,|\,1 - \exp\{[-K + 2ik(n-1)]\,s_l\}\,|^2, \tag{1}$$

$$\sigma_{\text{inel}} = \pi\lambda\!\!\!^{-}{}^2 \sum_{l=0}^{l\leqslant R/\lambda\!\!\!^{-}} (2l+1)\,[1 - \exp(-2Ks_l)], \tag{2}$$

$$s_l = (R^2 - l^2\lambda\!\!\!^{-}{}^2)^{1/2},$$

where $\lambda\!\!\!^{-} = k^{-1}$ is the wavelength of the incident mesons; $K = \lambda_t^{-1}$ is the absorption coefficient and n is the index of refraction of nuclear matter; R is the nuclear radius taken to be $1.4A^{1/3} \times 10^{-13}$ cm, which gives for carbon 3.2×10^{-13} cm. In order to estimate the error of the quasi-classical approximation, the cross sections were also calculated according to the exact quantum mechanical formulas:[5]

$$\sigma_{\text{el}} = \pi\lambda\!\!\!^{-}{}^2 \sum_{l=0}^{\infty} (2l+1)\,|\,\beta_l - 1\,|^2, \tag{3}$$

$$\sigma_{\text{inel}} = \pi\lambda\!\!\!^{-}{}^2 \sum_{l=0}^{\infty} (2l+1)\,(1 - |\,\beta_l\,|^2), \quad \beta_l = e^{2i\delta_l}, \tag{4}$$

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*The real part of the forward-scattering amplitude was obtained in Ref. 1 from dispersion relations.[3]

where δ_ℓ is the complex phase shift, obtained from the continuity condition imposed at the nuclear surface on the logarithmic derivative of the radial part of the wave function which satisfies the Klein-Gordon equation. For a square well, β_ℓ is expressed in terms of Bessel functions of half-integer order and arguments kR and κR where k and $\kappa = k + k_1 + iK/2$ denote the wave number of the incident particles outside and inside the well;

$$k + k_1 = nk = \frac{E - \mathrm{Re}\, V_0}{\hbar c}\left[1 - \frac{4\mu^2 c^4}{4(E - \mathrm{Re}\, V_0)^2 + (\hbar c K)^2}\right]^{1/2},$$

E is the total energy of the meson outside the nucleus and μ is the mass of the π meson. The results of the computation are presented in Figs. 1 and 2. The energy dependence calculated by Sternheimer[2] is also shown for purpose of comparison. In addition, the corresponding experimental values obtained from Refs. 6 — 15 are also shown on the figures. It may be seen from the figures that at large energies, the cross section calculated from formulas (1) and (2) is about 20 — 25% higher than that computed from (3) and (4). The curve showing the energy dependence of the inelastic cross section, as calculated by formulas (4), is in satisfactory agreement with experimental data. On the other hand, the elastic cross section which we have computed is considerably higher than the experimental data at meson energies below 100 Mev. Since the elastic scattering cross section increases rapidly with the depth of the potential well $|\mathrm{Re}\, V_0|$ at these energies, the presence of a sharp maximum in the cross section around 65 Mev, which does not agree with experiment, attests to the fact that the potential well depth used in the computations is too large in this energy region. A comparison of empirical estimates of Re V_0 with values computed in Ref. 1 is similarly not opposed to such a conclusion.

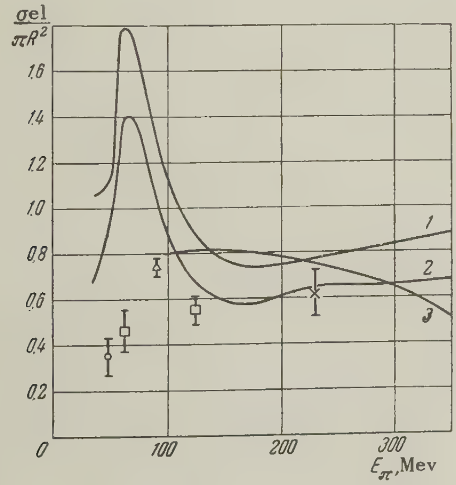

FIG. 1. Cross section for elastic scattering of π mesons by carbon nuclei as a function of energy. Curve 1 was calculated according to formula (1), Curve 2 according to formula (3), Curve 3 is taken from Ref. 2. □ - Refs. 6 and 7; ○ - Ref. 9; × - Ref. 13; Δ - Ref. 15.

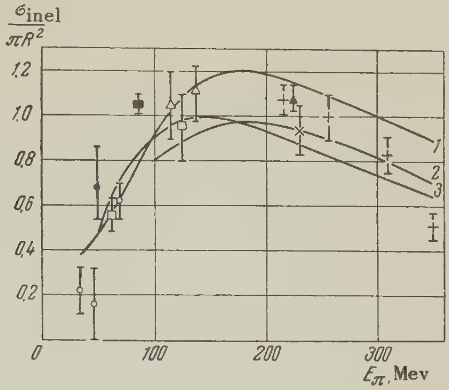

FIG. 2 Cross section for inelastic interaction of π mesons with carbon nuclei as a function of energy. Curve 1 was calculated according to formula (2), Curve 3 according to formula (4), Curve 2 is taken from Ref. 2 ○ - Ref. 8; ● - Ref. 9; □ - Ref. 6 and 7. ■ - Ref. 10; Δ - Ref. 11; Δ - Ref. 12; × - Ref. 13; + - Ref. 14.

Lack of experimental data on elastic scattering does not allow any more definite conclusions on the agreement between experimental and calculated results. In computing the cross sections it was also assumed that the nucleus is a sphere of constant density with a sharp surface. This is a crude approximation, at least for light nuclei. It has been shown[20] that the charge density decreases from the center to the surface. Calculations of the angular distribution of 62-Mev π mesons elastically scattered by carbon nuclei using a diffuse surface model[21] are in satisfactory agreement with

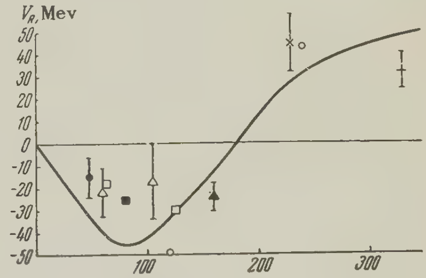

FIG. 3. Energy dependence of the real part Re V_0 of the optical-model potential for interaction of π mesons with complex nuclei according to Ref. 1 (solid curve). The estimates of the potential depth were obtained from experiments with carbon nuclei: □ - Refs. 6 and 7; ● - Ref. 9; × - Ref. 13; with aluminum nuclei: ■ - Ref. 16; with helium nuclei: Δ - Ref. 17; + - Ref. 18; with photoemulsion nuclei: Δ - Ref. 19. The symbol ○ denotes the maximum and minimum values of the potential obtained in Ref. 2.

experiment.[7] We therefore believe that computations of the total cross section for a diffuse surface nucleus is of definite interest.

In conclusion we take this opportunity to thank I. A. Kropin and B. F. Chumakov for carrying out a large part of the numerical calculations.

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33

RADIATIVE CORRECTIONS TO BREMSSTRAHLUNG

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FEYNMAN'S method,[1] which is usually used to obtain radiative corrections, cannot be applied in practice to bremsstrahlung because of the much greater calculational difficulties than those encountered, for instance, in the application to the corrections to Compton scattering.[2] In the present work, we use the so-called mass-operator method,[3,4] which has many advantages over the Feynman method.

The expression for the renormalized mass operator to the necessary order in e^2 has been obtained by Newton.[4] Not only the mass operator, but also vacuum polarization contributes to the radiative corrections to bremsstrahlung. In order to avoid the infrared catastrophe, as usual, we make use of the fictitious photon mass λ. We eliminate λ from the final expression by adding to the usual bremsstrahlung cross section the cross section for double bremsstrahlung, when two photons are emitted simultaneously and the energy of one of these photons is less than some quantity ΔE determined by the accuracy of the measurement.

The total cross section is conveniently written in the form

$$d\sigma \doteq d\sigma_0 [1 - (e^2/\pi)(\delta_R + \delta_D)], \quad e^2 = 1/137,$$

where $d\sigma_0$ is the cross section for the basic process and is given by the Bethe-Heitler formula,[5,6] δ_R gives the radiative corrections, and δ_D gives the double bremsstrahlung.

Exact expressions for δ_R and δ_D, together with a description of the method by which they have been calculated, will be presented in a detailed article. Here we shall give only some limiting values of δ_R and δ_D. We choose units in which $\hbar = c = m = 1$, and make use of the following notation: p_1 and ϵ_1 are the initial, and p_2 and ϵ_2 are the final electron momentum and energy; k and $\omega = \epsilon_1 - \epsilon_2$ are the momentum and energy of the emitted photon;

$$\varkappa = -2\omega(\varepsilon_2 - p_2 \cos\theta_2), \quad \theta_2 = \widehat{\mathbf{kp}_2};$$

$$\tau = 2\omega(\varepsilon_1 - p_1 \cos\theta_1), \quad \theta_1 = \widehat{\mathbf{kp}_1};$$

$$\alpha = \varkappa + \tau;$$

$$\rho = -2 + \alpha + 2(\varepsilon_1\varepsilon_2 - p_1p_2\cos\theta), \quad \theta = \widehat{\mathbf{p}_1\mathbf{p}_2};$$

$$4\sinh^2 x = \rho, \quad 4\sinh^2 y = \rho - \alpha; \quad xh(x) = \int_0^x u\coth u\, du.$$

1. The limiting case of low frequencies ($\omega\epsilon_1 \ll 1$). In this case

$$|\varkappa|, \tau \ll 1, \quad \rho \approx 4p^2\sin^2(\theta/2);$$

$$\delta_R = 2[1 - 2x\coth(2x)](\ln\lambda + 1) + x\tan x + 4x\coth(2x) \times [h(2x) - h(x)] + \frac{2x}{\sinh(2x)}\frac{\rho}{4\varepsilon^2 - \rho} + w(x) + O(\omega\varepsilon_1); \quad (1)$$

$$\delta_D = 2[1 - 2x\coth(2x)]\ln\frac{2\Delta E}{\lambda} + \frac{1}{v}\ln\frac{1-v}{1+v} + \frac{(1-v^2)\cosh(2x)}{v\sin\theta/2}G(v,\theta) + O(\omega\varepsilon_1); \quad (2)$$

where

$$G(v,\theta) = \int_{\cos(\theta/2)}^{1}\frac{du}{(1-v^2u^2)(u^2-\cos^2(\theta/2))^{1/2}}\ln\frac{1+vu}{1-vu};$$

$$p_1 \approx p_2 = p, \quad \varepsilon_1 \approx \varepsilon_2 = \varepsilon, \quad v = p/\varepsilon.$$

Here and below we use

$$w(x) = 2(1 - x\coth x)(1 - {}^1/_3\coth^2 x) - {}^2/_9$$

to denote the contribution from vacuum polarization. Expressions (1) and (2) coincide, as may have been expected, with the radiative corrections to elastic scattering as calculated by Schwinger[7] (see also Akhiezer and Berestetskii,[6] Sec. 45).

2. The relativistic case (ϵ_1 and $\epsilon_2 \gg 1$) with low energy losses ($\omega \ll \epsilon_1$ and ϵ_2) at small angles (θ, θ_1, and $\theta_2 \sim 1/\epsilon \ll 1$; we note that at high energies it is just such small angles that give the main contribution). In this case

$$|\varkappa| = -\varkappa = \omega/\varepsilon_2 + \omega\varepsilon_2\theta_2^2 \ll 1,$$

$$\tau = \omega/\varepsilon_1 + \omega\varepsilon_1\theta_1^2 \ll 1, \quad \rho = \varepsilon_1\varepsilon_2\theta^2 \sim 1;$$

$$\delta_R = 2[1 - 2x\coth(2x)](\ln\lambda + 1) + x\tanh x + 4x\coth(2x) \times [h(2x) - h(x)] + w(x) + O(\varkappa, \tau); \quad (3)$$

$$\delta_D = 2[1 - 2x\coth(2x)]\ln\frac{2\Delta E}{\lambda} - 4x\coth(2x)[h(2x) - h(x)] + 2\ln(2\varepsilon)[2x\coth(2x) - 1] + O\left(\frac{\omega}{\varepsilon}\ln\varepsilon, \frac{\omega}{\varepsilon}\right). \quad (4)$$

It should be noted that cases 1 and 2 overlap in the region ϵ_1, $\epsilon_2 \gg 1$; $\omega\epsilon_1 \ll 1$; θ, θ_1, $\theta_2 \sim 1/\epsilon \ll 1$), and that in this region Eqs. (1) and (2) become (3) and (4).

3. In the extreme relativistic limit, when

$$\varepsilon_1, \varepsilon_2, \omega \gg 1; \quad \ln(\rho - \alpha), \quad \ln\rho, \quad \ln|\varkappa|, \quad \ln\tau \gg 1,$$

but

$$\ln\frac{\rho-\alpha}{|\varkappa|}, \quad \ln\frac{\rho-\alpha}{\tau}, \quad \ln\frac{\rho}{|\varkappa|}, \quad \ln\frac{\rho}{\tau} \sim 1,$$

the quantities δ_R and δ_D are

$$\delta_R = 2(1 - 2y)\ln\lambda + 2y^2 - 3y + {}^4/_3 x + O(1); \quad (5)$$

$$\delta_D = 2(1 - 2y)\ln\frac{2\Delta E}{\lambda} + 2y^2 - 2y\ln\frac{\rho - \alpha}{4\varepsilon_1\varepsilon_2} - \ln\varepsilon_1\varepsilon_2 + O(1), \quad (6)$$

where

$$2x = \ln\rho, \quad 2y = \ln(\rho - \alpha).$$

In the nonrelativistic limit

$$\delta_R = {}^2/_3\{-(\mathbf{p}_1 - \mathbf{p}_2)^2(\ln\lambda + {}^1/_5) + 2(\mathbf{p}_1 - \mathbf{p}_2)\mathbf{k}\ln\omega\} + O(p^3); \quad (7)$$

$$\delta_D = {}^2/_3(\mathbf{p}_1 - \mathbf{p}_2)^2\left(\frac{5}{6} - \ln\frac{2\Delta E}{\lambda}\right) + O(p^3); \quad (8)$$

$$2\omega = p_1^2 - p_2^2.$$

In the limit as $\omega \to 0$ when $\mathbf{p}_1 = \mathbf{p}_2 = \mathbf{p}$ these expressions lead to the nonrelativistic limit of the corrections to elastic scattering,[7] namely

$$\delta_R + \delta_D = \frac{8}{3}p^2\sin^2\frac{\theta}{2}\left[\frac{19}{30} - \ln(2\Delta E)\right].$$

From (7) and (8) it follows that in the limit as $p_1 \to 0$, the corrections vanish.

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34

EQUATION OF STATE FOR SOLID ARGON

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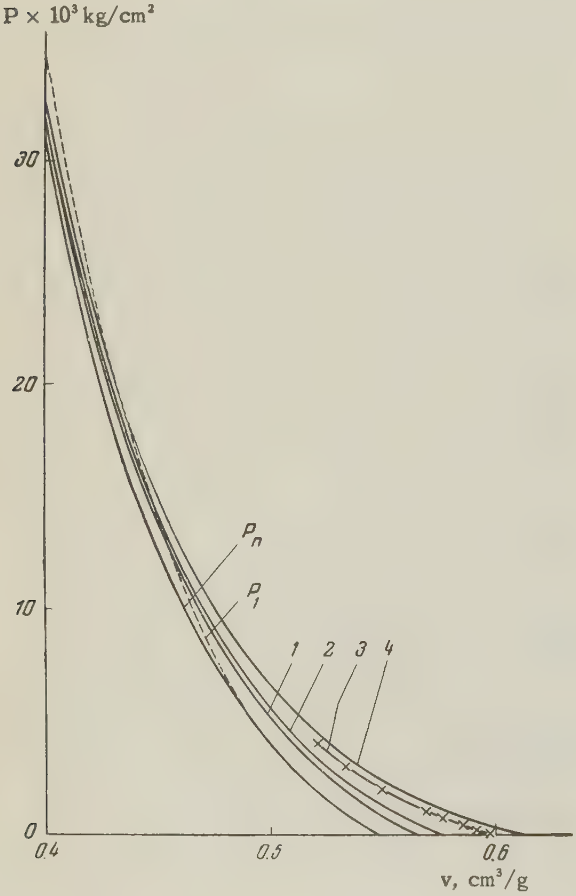
LET us take the volume dependence of the energy of a molecular crystal to be of the form[1]

$$E = A\exp(-Bx^{1/3}) - Cx^{-2}, \tag{1}$$

where A, B, and C are experimentally determined constants, and $x = \rho_0/\rho$ is the reduced volume, ρ_0 being the density at absolute zero.

The power-law potential $E_1 = A_1x^{-3} - C_1x^{-2}$ used by Domb[2] is less accurate than Eq. (1), especially at high pressures.[1] This can be seen from the figure, where the dashed curve P_1 was obtained from Domb's potential E_1, and curve P_n was obtained from formula (1).



Pressure – volume relationship for crystalline argon. 1 – T = 65°K. 2 – T = 80°K. The crosses denote experimental points at 65°K taken from the work of Stewart.[3]

The free energy per unit mass, according to Debye's theory of the crystalline state, has the form

$$F = A\exp(-Bx^{1/3}) - Cx^{-2} + \frac{9}{8}Nk\Theta + NkT\,[3\ln(1 - e^{-\Theta/T}) - D(\Theta/T)] \tag{2}$$

where N is the number of atoms per unit mass, k is Boltzmann's constant, Θ is Debye's characteristic temperature, which depends on the volume, T is the absolute temperature, and D(Θ/T) is the Debye function.

The third term in (2) is the zero-point vibrational energy of the lattice atoms, which for argon amounts to about 10% of the total energy of the crystal.

Using Eq. (2), we find the pressure P, the isothermal compressibility κ_T, and the coefficient of thermal expansion α from the formulas:

$$P = -\rho_0\left(\frac{\partial F}{\partial x}\right)_T;\ \frac{1}{\varkappa_T} = -x\left(\frac{\partial p}{\partial x}\right)_T;\ \alpha = \frac{1}{x}\left(\frac{\partial x}{\partial T}\right)_P = \varkappa_T\left(\frac{\partial P}{\partial T}\right)_x.$$

To determine the parameters A, B, and C we have used two points on the experimental curve P(x) at T = 65° K,[3] the value of the density ρ_0 from Dobbs et al,[4] and the expression $\Theta = ax^{2/3} \times (d^2F/dx^2)^{1/2}$, which follows immediately from the method of determining Θ when the constant a is found from the Debye temperature for T = 0.[5] Then with some reasonable approximations we obtain the following values for the constants:

$$A = 8.000;\quad B = 13.078;\quad C = 3.877 \text{ cal/mole}.$$

These values were used to calculate the parameters ρ(T) and α(T) shown in the table, and P(v) for 65° K as shown in the figure, where v = 1/ρ. It can be seen that they agree well with the experimental results. The same good agreement is also found between the calculated and experimental values for the adiabatic compressibility.

$T°$ K		20	40	60	80
ρ, g/cm³		1.763	1.735	1.690	1.634
	experiment[4]	1.764	1.737	1.691	1.636
α × 10⁴, deg⁻¹		5.6	10.7	14.8	19.4
	experiment[4]	4	12	15	18

From this it follows that the potential formula (1), when used with the above values of the parameters, is a good description of the current experimental data for solid argon, and may be used for extrapolating the equation of state to higher pressures.

In conclusion, I wish to thank V. N. Zharkov for proposing and discussing this subject.

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35

TWO CASES OF HYPERFRAGMENT DECAY

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UP to now only several decays of hyperfragments with $Z \geq 6$ has been described in the literature.[1-5] In three cases the binding energy of the Λ^0 particle has been estimated:

$$^{\mathrm{C}}B_{\Lambda} = (13 \pm 6), \quad ^{\mathrm{N}}B_{\Lambda} = (20 \pm 11),$$

$$^{\mathrm{O}}B_{\Lambda} = (19.0 \pm 14.0)\ \text{Mev}.$$

In a stack of Ilford G5 emulsions, exposed in the stratosphere, we have observed a heavy hyperfragment which decayed by emission of a fast proton. This hyperfragment was interpreted as being F_{Λ} or Ne_{Λ}. A mesonic decay of a hyperfragment was also observed in this stack.

Case 1. A multiply charged hyperfragment is emitted from a star of the type 15 + 2n with a range $R = 127\mu$. The absence of δ rays near the end of the track and the narrowing of the track indicate that the hyperfragment came to rest. The charge of the hyperfragment is estimated from the length of the narrowing to be $Z = 8 \pm 2$. At the end of its track the hyperfragment decays into three charged particles (a, b, c in Fig. 1).

Particles a and b stop in the same emulsion after travelling 204 and 20μ respectively. From the number of gaps and of δ rays, the charges of these particles were determined to be 1 and 1 – 2 respectively.

Particle c leaves the stack after penetrating 8 emulsions. The grain density and the multiple scattering yield a mass of $(2170 \pm 300)\ m_e$. Particle c thus can be identified to be a proton. Its energy is (117.3 ± 12.4) Mev.

The residual momentum of the three charged particles was computed for all possible types of particles allowed for a and b. Assuming that no neutral particles participate in the decay, the momentum unbalance has to be taken up by recoil of the residual nucleus leading to a range $R \leq 0.8\mu$.

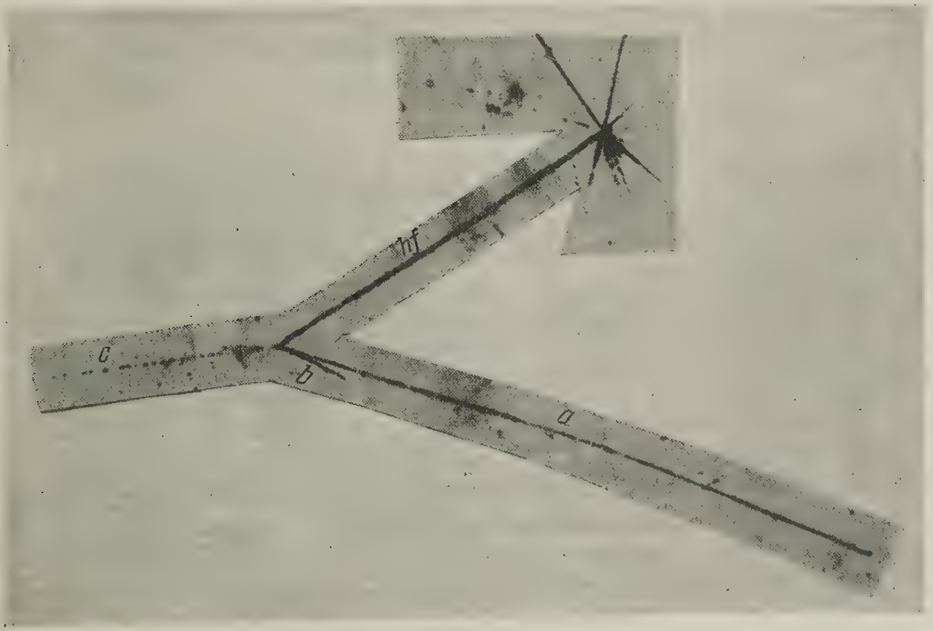

FIG. 1. Decay of a heavy hyperfragment.

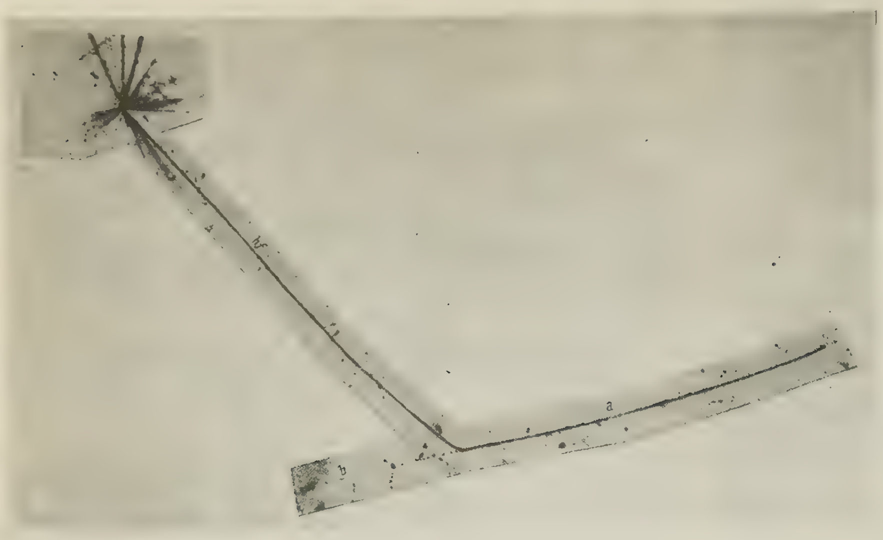

FIG. 2. Mesonic decay of a hyperfragment.

(A range R = 0.8μ is the largest range a particle can have without producing a visible track.)

The binding energy of the Λ^0 particle was determined for all possible decay schemes and those leading to positive binding energies were selected. This analysis leads to the following decay schemes of the hyperfragment:

$$ {}_\Lambda F^{18,19,20} \to d\,(t) + p\,(d,t) + p + C, \qquad (1)$$

$$ {}_\Lambda Ne^{20,21} \to p\,(d,t) + p(d,t,He^3,He^4) + p + N\,(C) \qquad (2)$$

with the following values for the binding energy: for decay scheme (1), ${}^F B_\Lambda = (18.9 \pm 16.3)$ Mev; for (2) ${}^{Ne}B_\Lambda = (21.8 \pm 17.7)$ Mev.

If a neutral particle participated in the decay the possibility is not excluded that the hyperfragment was actually lighter than F.

Case 2. A light hyperfragment is emitted from a star of type 21 + 8p. After travelling 276μ it decays into two particles (Fig. 2). The scattering of the hyperfragment indicates a decay at rest. From the number of gaps and of δ rays its charge was determined to be $Z = 2-3$.

Particle a has a range of 218μ and was identified to have a charge 1. Track b leaves the stack after traversing (12,320 ± 500) μ in 13 emulsions.

A comparison of the track density with calibration curves showed that the track was due to a π meson with an energy of (32.8 ± 5.0) Mev.

A kinematical analysis yields the schemes

$$ He^5_\Lambda \to p + \pi^- + He^4 + Q_1, \qquad (3)$$

$$ Li^{7,8}_\Lambda \to p + \pi^- + Li^{6,7} + Q_2, \qquad (4)$$

with $Q_1 = Q_2 = (39.0 \pm 5.0)$ Mev.

Taking $Q_\Lambda = 36.9$ Mev[6] we obtain $B_\Lambda = (-2.1 \pm 5.0)$ Mev.

In conclusion the authors express their thanks to A. I. Alikhanian for his interest in the work and to V. M. Kharitonov for valuable comments.

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36

THE EFFECT OF COULOMB CORRELATIONS ON THE SPECTRUM OF ELECTRON-PLASMA OSCILLATIONS

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THE numerous articles on the collective oscillations of plasmas have thus far, to our knowledge, not clarified the role of Coulomb correlations. This problem can be approached in several ways. For example, a kinetic equation which takes the correlations into account can be used to derive the dispersion equation for small density oscillations. This dispersion equation would express the oscillatory frequency $\omega(\mathbf{k})$ in terms of parameters that characterize a uniform spatial distribution (the ground state), the principal characteristic being the average energy of a single particle when all interactions (kinetic energy, exchange interaction, and Coulomb correlations) are taken into account.

The second possible method was considered in Ref. 1. Small deviations from uniform density are described by linearized time-dependent Hartree equations in hydrodynamical form, with the coefficients of the equations given in terms of the ground-state particle energy. In Ref. 1 the dispersion equation for weak excitations (long waves) is given as

$$\omega^2 = n_0 G(k) k^2 / m + 2\bar{E}_0 k^2 / m + O(k^4),$$

where $G(\mathbf{k})$ is a Fourier component of the interaction potential, $\bar{E}_0$ is the average energy per particle taking into account exchange and Coulomb correlations in the ground state, and n_0 is the average particle density.

For an electron gas we have

$$\bar{E}_0 = E_F + E_x + E_C,$$

where $E_F = 3P_0^2/10m$ is the Fermi kinetic energy, $E_x = -(3e^2/4\pi\hbar)P_0$ is the Coulomb exchange energy, and E_C is the Coulomb correlation energy. Wigner's calculations[2] give

$$E_C = -0.288/(r_s + 5.1\, r_B), \qquad r_s = (3/4\pi n_0)^{1/2},$$
$$r_B = e^2/m\hbar^2.$$

This expression for E_C is valid with 20% accuracy for both small and large densities. Transforming to the usual notation of the theory of electron-plasma oscillations, we obtain the following expression for the frequency of normal oscillations:

$$\omega^2 = \omega_0^2 + \left[1 - 0.47\xi^2 - \frac{0.065}{1 + 0.22\xi^2}\xi^4\right]\left(\frac{3P_0^2}{5m^2}\right)k^2 + O(k^4);$$

here we have ω_0 for the Langmuir frequency, and the dimensionless quantity $\xi = \hbar\omega_0/(P_0^2/2m)$. For metals $\xi \sim 1$, and exchange effects play a larger part than Coulomb correlations, whose contribution in the dispersion equation is about seven times smaller. For a very dense electron gas ($r_s/r_B \ll 1$) the correlation energy was recently calculated by Gell-Mann and Brueckner,[3] who obtained the following expression:

$$E_C = [0.0622 \ln(r_s/r_B) - 0.096](e^4 m/2\hbar^2).$$

Here the contribution of Coulomb correlations to the dispersion equation is still smaller. This can easily be understood, because in a dense gas the Fermi kinetic energy is proportional to $n_0^{2/3}$ and the energy associated with interactions is proportional to $n_0^{1/3}$. Therefore as the density increases the kinetic energy rises more rapidly than the Coulomb interaction energy.

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37

FILM TRANSFER RATE IN HELIUM ISOTOPE MIXTURES

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As is well-known,[1,2] the effect of He^3 dissolved in He II upon the phenomenon of film transfer is to reduce the transfer rate.

It appeared of some interest to investigate this situation in more detail, for which purpose experiments were carried out with helium isotope mixtures of 1.5, 4.7, 7.0, and 9.6% He^3 content. The experiments were performed with the aid of an apparatus consisting of two reservoirs, communicating through the helium film, made of thin-walled capillary tubing (1.08 mm in diameter) and of different lengths,[1] so that as the helium isotope mixture was condensed into the system a difference in level was established, and film transfer from the higher level to the lower began to take place below the λ-point. Measurements of the rate of change in the height of the level in one of the reservoirs of the system, made with the aid of a cathetometer, permitted determination of the film transfer rate $R = v\delta$, where v is the velocity of the film and δ is its thickness. The temperature region investigated was immediately adjacent to the λ-point. This is due to the fact that at sufficiently low temperatures (where the transfer rate becomes large) a substantial difference in the concentrations of the mixtures in the two reservoirs of the system may develop, and the resulting osmotic pressure (cf. Ref. 1) may seriously affect the results obtained.

The results obtained are given in Figs. 1 and 2, in the first of which is shown the dependence of the

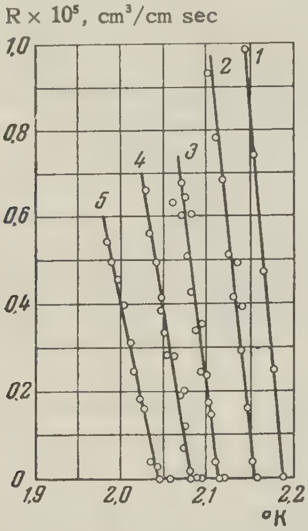

FIG. 1. Temperature dependence of the film transfer rate, for helium isotope mixtures having He^3 concentrations of 1.5% (curve 2), 4.7% (curve 3), 7.0% (curve 4) and 9.6% (curve 5). Curve 1 gives the same dependence for pure He^4, obtained by extrapolation of the curves showing the concentration dependence of the film transfer rate.

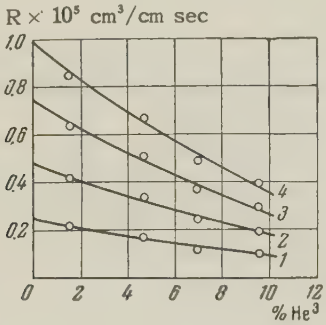

FIG. 2. Dependence of the film transfer rate upon He^3 concentration. Curve 1 refers to $T_\lambda - T = 0.01°$, 2 – 0.02°, 3 – 0.03°, 4 – 0.04°.

film transfer rate upon temperature, and in the second, upon the He^3 concentration. It follows immediately from these diagrams that the film transfer rate is reduced with increasing He^3 concentration.

Comparing the data on the temperature dependence of the normal component density for helium isotope mixtures,[3,4] one may conclude that in the temperature region investigated the film transfer rate is directly proportional to the density of the superfluid component: $R = A\rho_S/\rho$, where $A = 3.2 \times 10^{-5}$ cm³/cm sec.

It should be noted that the temperature of the He I – He II phase transition could be determined for the mixtures used from the onset of the film flow. In this way values were obtained which were in convincing agreement with similar data obtained by other methods.[5]

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Translated by S. D. Elliott
38

RELATION BETWEEN SECONDARY EMISSION OF NEGATIVE IONS AND THE ANGLE OF ENTRY OF PRIMARY PROTONS INTO A METAL TARGET

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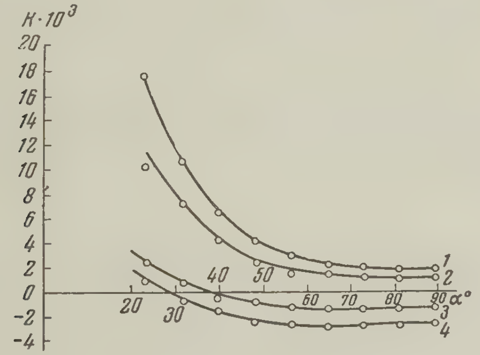
IN an earlier paper[1] we have determined the coefficient for K-emission of secondary negative ions from a target under bombardment by hydrogen ions with energies of 200 – 1,000 kev, and have shown that the total coefficient for secondary emission of negative ions, K^-, does not exceed 10^{-3}. However, it might be supposed that at small angles of entry of the proton beam into the target the coefficient K^- would increase, as in the case in secondary electron emission.[2]

In the present work we have attempted to estimate the change in the coefficient K^- as a function of the entry angle of the proton beam striking the target. To carry out this investigation we used a proton beam with an energy of 50 kev, extracted by a magnetic analyzer; the method described earlier[1] was used with the following modifications: the target, which was the internal electrode of a spherical condenser, was a circular plate fastened to a hemisphere 28 mm in diameter, inside of which there was a tungsten helix for heating the target. The entire spherical condenser could be rotated about an axis in the plane of the target. As in the earlier work, the spherical condenser was located in a magnetic field directed along the axis of rotation of the target. The magnetic field required for suppression of electron emission was less than 100 oersted. A retarding potential of 400 volts was applied to the target to suppress the emission of slow secondary positive ions. All experiments were carried out in a vacuum of 2×10^{-6} mm Hg.

The relation obtained between the coefficient of secondary negative ion emission and the angle of entry of the beam are shown in the figure. As is apparent from the figure, in copper and E Ia–1 stainless steel this coefficient is positive over the entire entry-angle region which was studied. In aluminum and beryllium targets the secondary ion emission coefficient is negative at large entry angles; at entry angles below 30 – 40° it passes through zero and becomes positive.



The secondary ion emission coefficient as a function of entry angle of the primary proton beam in a plane target. 1 – Cu, 2 – stainless steel, 3 – Al, 4 – Be.

Preheating the targets at 900° C for 20 minutes results in a reduction of K^- in the beryllium targets and an increase in K^+ in the copper targets.

The present results can be understood if one keeps in mind the fact that the secondary-ion emission is composed of true secondary negative ions and of protons of the primary beam, scattered at angles larger than 90° by the Coulomb field of the target nuclei. The observed sign of the secondary-emission coefficient depends on the relative strengths of these two components.

The results obtained indicate that at proton energies of 50 kev in copper and stainless steel targets the number of scattered protons is larger than the number of secondary negative ions at all entry angles. In aluminum and beryllium targets the number of secondary negative ions exceeds the number of scattered protons at entry angles below 30 – 40°.

Attempts to compute the current of single positive scattered ions indicate that the calculation errors are as large as the experimental errors; hence it is impossible to determine accurately the negative-ion coefficient from the difference of currents when the present method is employed.

The authors wish to express their gratitude to A. K. Val'ter, Ia. M. Fogel', and E. V. Inopin for a discussion of the results and interest in the work.

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39

CYCLOTRON RESONANCE IN LEAD AT 8,900 Mcs

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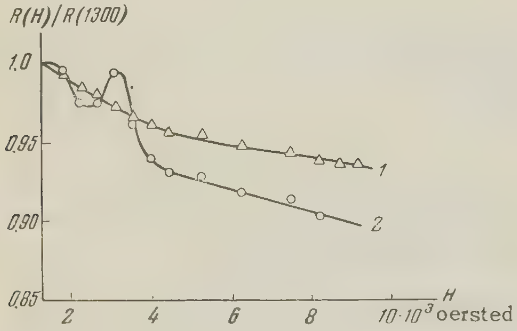
CYCLOTRON resonance effects in metals, predicted theoretically by Azbel' and Kaner,[1] have been recently observed experimentally in tin[2,3] and bismuth.[4]

In the present note we present a brief report concerning the results of experiments on cyclotron resonance in lead at 8,900 Mcs.

The choice of lead as a test material is occasioned by the following considerations. It follows from the theory of cyclotron resonance that the amplitude of the oscillations of surface resistance of the metal in the resonance region at a given frequency of the alternating electromagnetic field depends in an important way on the electron relaxation time in the metal t_0, increasing as t_0 increases. In contrast with the materials investigated earlier, such as copper and tin,[2,3] for which the dc resistance changes slowly in the helium-temperature region, lead has the feature that its conductivity in the helium temperature region increases significantly as the temperature is reduced,[5] indicating an increase in the electron relaxation time t_0. Thus, the change due to resonant effects, observed in the same sample of tin as the temperature was reduced from 4.2° K to 2° K,[3] should have a larger effect in lead, by virtue of the observations given above.

The sample was a lead single-crystal wire approximately 12 mm in length and approximately 0.8 mm in diameter extracted in a quartz capillary from lead obtained from the Kalbaum company;* the sample was placed along the axis of a coaxial copper resonator. The measurement of the surface resistance of the sample was carried out by the same methods used in studying cyclotron resonance in tin.[3]

The results of the measurements of $R(H)/R(1300)$ [$R(H)$ is the surface resistance in a fixed field $H > 1300$ oersted†] in lead at a frequency of 8,900 Mcs and temperatures of 4.2° K (Curve 1) and 2° K (Curve 2) are shown in the figure. The effect of electron relaxation time is obvious from the figure. At 4.2° a monotonic decrease of resistance is noted with increasing field; at a temperature of 2° K and $H \approx 2400$ oersted there is rather deep resonance minimum followed by a maximum; then the surface resistance falls off sharply in accordance with the theoretical predictions. An estimate of the effective mass m^* of the conduction electrons in lead, carried out under the assumption that the resonance minimum at $H = 2400$ oersted corresponds to the condition $\nu = eH/2\pi m^* c$, yields the value $m^* = 0.8\ m_0$.



The fact that the obtained effective conduction-electron mass does not differ significantly from the free-electron mass indicates that just as in tin,[3] the electrons at the bottom of the band are responsible for the cyclotron resonance in lead.[6]

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*The same samples of lead were investigated by Borovik.[5]

†In view of the fact that lead is a superconductor with $T_c = 7.2°$, the impedance in the field does not refer to the value of R at H = 0, but to its value at 1300 oersted, when the sample is in the normal state

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40

INVESTIGATION OF THE SURFACE RESISTANCE OF TIN IN WEAK MAGNETIC FIELDS

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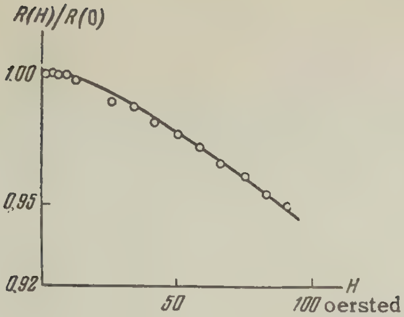
In work on cyclotron resonance[1] it has been shown that in weak fields parallel to the surface of the metal the surface resistance is a very weak function of the field. On the other hand, experiments carried out by Fawcett[2] on cyclotron resonance in tin and copper have shown that in weak fields there is a considerable dependence of surface resistance on field; $dR(H)/dH$ is considerably different from zero as $H \rightarrow 0$. The same conclusion may be drawn from our earlier work.[3]

In this connection we have investigated the surface resistance of tin in fields up to 100 oersted, using the method escribed earlier.[3] The results of these experiments, carried out with the tin sample used earlier[3] at a frequency of 9,300 Mcs and a temperature of 4.2° K are shown in the figure. These results indicate that, in accordance with the theoretical predictions, the real part of the resistance of the metal is virtually independent of



field up to 10 oersted, i.e., $dR/dH \rightarrow 0$ as $H \rightarrow 0$. In fields with $H > 10$ oersted there is a monotonic reduction of the surface resistance with increasing field, in agreement with the results of the earlier work.[2,3]

It is interesting to note that the dependence of the surface resistance of metals on the magnitude of the magnetic field should be taken into account in measuring the temperature dependence of R_s/R_n for superconductors (R_s and R_n are the surface resistances of the metal in the superconducting and normal state) if as R_n we take the surface resistance of the metal under conditions in which the superconductivity is destroyed by the field. Neglecting this effect may lead to values of R_s/R_n which are too high.

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41

THE PHASE DIAGRAM OF THE HYDROGEN-DEUTERIUM SYSTEM

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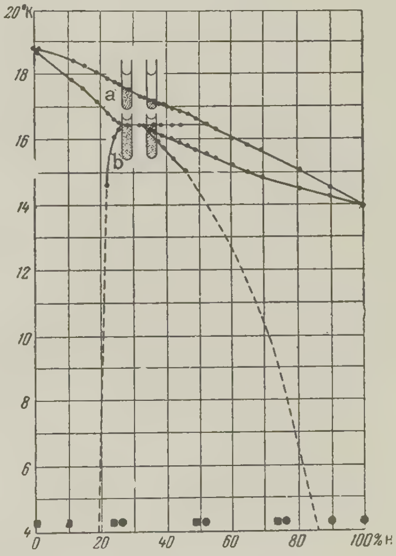
In a previous letter[1] we reported some preliminary results of x-ray studies of hydrogen and deuterium at liquid helium temperatures. In it we showed that the previous determinations of the crystal structure of hydrogen at Leyden[2] had been done incorrectly.

At the Symposium on Low Temperature Physics in Leningrad in June, 1956, we also reported some results of crystal structure studies on mixtures of the hydrogen isotopes, in which it was shown that there was only a limited range of concentrations for which a solid solution could exist in this system.

In the present report we shall present more detailed data for the system, obtained from the thermal analysis of hydrogen-deuterium mixtures. The mixtures were prepared from the pure isotopes, condensed into a calorimeter cooled with liquid hydrogen kept at reduced pressure by a vacuum pump, and then frozen. After the pump was stopped, the mixture was allowed to warm up slowly over the temperature interval from 14° to 19° K.

Thermal analysis revealed a horizontal portion in the solidus curve at a temperature of 16.4° K. Comparison of the thermal analysis data with the results of x-ray patterns at 4.2° K enabled us to determine the approximate limits of the two-phase region, to connect this low-temperature portion with the solidus line, and outline the general shape of the phase diagram for the hydrogen-deuterium system (see the figure).

During the crystallization of mixtures in the concentration range 26 — 52% of hydrogen by volume, the presence of peritectic regions was established visually. We made use of the fact that in the vapor over liquid helium a stable tempera-



○ – results of thermal analysis, ■, ● – results of x-ray analysis (■ – D_2 lattice; ● – H_2 lattice) ture gradient can be set up which will persist for some time. A set of two glass test tubes, containing condensed mixtures whose composition lay within the concentration limits given above, were placed in a Dewar flask over liquid helium. The test tubes were so placed that the level corresponding to the temperature of onset of crystallization passed through them. The boundary between the solid and liquid phases established itself at different levels in mixtures of different composition, being at a lower level in mixtures richer in hydrogen (see a in the phase diagram, which is a schematic representation of the test tubes). The boundary could be caused to shift by moving the test tubes to a region of lower temperature. As the amount of the solid phase increased, the liquid phase became increasingly richer in hydrogen, and the boundary in tubes with hydrogen-rich mixtures overtook the boundary in tubes containing more deuterium, corresponding to the equalization of liquid-phase concentrations when the peritectic region is reached. After the solid-liquid boundary had reached the same level in both tubes (see b in the diagram), further crystallization proceeded at the same temperatures in both tubes.

In addition to the thermal analysis, an x-ray study of the hydrogen-deuterium mixtures, and of the pure isotopes, was made. Some improvements in the method of taking the photographs made it possible to eliminate the parasitic lines which were present in the patterns of previous work, as has been explained. As a result, two lines were found to be present due to the hydrogen lattice, corresponding to spacings $d = 3.15$ A and $d = 2.79$ A; but from the deuterium lattice there was only one, corresponding to $d = 2.84$ A. No lines were found at wide angles, because of the rapid falling off of the scattered intensity. The x-ray patterns from mixtures with hydrogen concentrations lying in the interval 20 to 80% contained lines from both the hydrogen and the deuterium lattices, with parameters only slightly altered. This confirms the conclusion we have expressed already, that the structure of hydrogen was incorrectly determined by the workers at Leyden, and that there is a region of concentration in which hydrogen and deuterium exist as a mixture of two solid phases. As to the accurate determination of the structures of hydrogen and deuterium, the question must remain open for the time being, because of the difficulty of assigning definite indices to x-ray patterns which contain so few lines. It is, however, possible to state unequivocally that they do have different lattices.

The results which we have obtained, indicating a separation of the solid mixture of hydrogen isotopes into two phases, agree with the conclusions of Prigogine et al.[3] that there is a crticial temperature below which a solid solution of isotopes must separate. For the hydrogen isotopes, however, these authors estimated a critical temperature below 1° K; I. Lifshitz and Stepanova[4] have shown that the critical temperature for separation is equal to $T = \epsilon^2 \Theta_D$ ($\epsilon = \Delta m/m$ and Θ_D is the Debye temperature); i.e., for hydrogen isotopes it is of the same order as their melting point. This agrees completely with the shape of the phase diagram for the hydrogen-deuterium system, as reported in this letter.

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Translated by D. C. West
42

CORRECTION TO THE ARTICLE "ON THE STRUCTURE OF THE ELECTRON SPECTRUM IN LATTICES OF THE TELLURIUM TYPE"

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IN this paper[1] the conclusion was drawn, from the relations (22) and the fact that Λ_2 and Λ_3 are complex conjugate quantities, that the terms E_2 and E_3 are in contact along the entire k_z axis. It is, however, easy to convince oneself that for the "rotating" elements $U_2^{(j)}$ the equation

$$\sum_{j=1}^{3} \chi_j [(U_2^{(j)})^2] = 3, \tag{1}$$

holds. According to Herring[2] this indicates the absence of supplementary contact of the zones because of time reversal. This is due to the fact that only $\psi_{k_z,3}$ and $\psi_{-k_z,2}$ belong to a single value of the energy, and not $\psi_{k_z,3}$ and $\psi_{-k_z,3}$. The statements just made follow from the relations

$$\widetilde{C}_3^1\psi_{k_z,3} = \omega \exp\{ibk_z/3\}\psi_{k_z,3};$$

$$\widetilde{C}_3^1\psi_{-k_z,3} = \exp\{-ibk_z/3\}\omega^2\psi_{-k_z,2} \tag{2}$$

The functional dependence $E(k_z)$ for $k_x = k_y = 0$ can be found by the following considerations. It is well known that displacement by a vector of the reciprocal lattice takes the entire spectrum over into itself, i.e., we have the equations

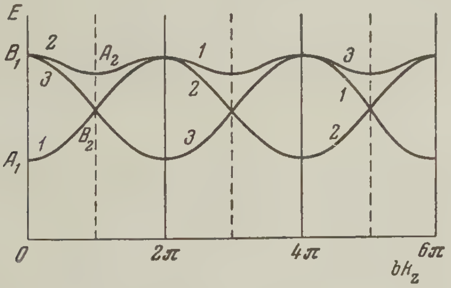
$$E(k_z + 2\pi/b) = E(k_z), \quad \psi_{n,k_z+2\pi/b} = \psi_{n,k_z}. \tag{3}$$

For $k_z \to 0$ we get the result that $U_{j,2\pi/b} = e^{\frac{2\pi i}{b}z}U_{j,0}$. Thus for the "open" element C_3^1 the laws of transformation of $U_{j,0}$ and $U_{j,2\pi/b}$ will be different:

$$\widetilde{C}_3^1U_{1,2\pi/b} = \omega^2U_{1,2\pi/b}; \quad \widetilde{C}_3^1U_{3,2\pi/b} = U_{3,2\pi/b};$$

$$\widetilde{C}_3^1U_{3,2\pi/b} = \omega U_{2,2\pi/b}. \tag{4}$$

But if we agree to characterize the n-th term of Eq. (1) of Ref. 1 by the transformation law $U_{n,k}$, then the term $E_1(2\pi/b)$ must be regarded as the continuation of $E_3(k_z)$ for $k_z \to 2\pi/b$, and so on. In the notations adopted, E_j will be periodic functions of k_zb with the period 6π.



The diagram shows the pattern of the term splitting, and indicates the course of the terms through the interval $0 \le k_zb \le 6\pi$ by the numbers 1, 2, 3. As follows from the diagram, the calculations of the dispersion law carried out in Ref. 1 for points of type Λ have meaning only in case of accidental crossing of the terms E_2 and E_3 (which is permissible in principle, because of their different symmetries). From considerations of symmetry one can show that the points A_1 and A_2 in the diagram are points of zero slope. But it makes sense to suppose that there can also be present additional points of zero slope, connected with the concrete form of the potential $U(\mathbf{r})$, as occurs in electronic germanium, in which the minimums are not located at the point $\mathbf{k} = 0$, although this is a point of zero slope. These assumptions impose the following limitations on the functions U_1 and U_2 that transform according to the representation Γ_3: $p^z_{1,2} = 0$. All the remaining quantities $p^x_{i,j} = p^y_{i,j} = p^z_{i,j} = 0$ by the selection rules (here $i, j = 1, 2$). Then the points B_1, B_2 (see diagram) can be extremal points. The functional dependence of $E(\mathbf{k})$ for the points Γ, A, K, H was obtained in Ref. 1 on just this assumption. It must be kept in mind that the contact of the bands along the z axis is maintained only in the approximation quadratic in k. All the results of Ref. 1 are correct with the reservations that have been indicated. As regards the valence band, at the point B_2 in the diagram the degeneracy is in all probability removed on account of the spin-orbit interaction; otherwise for the four p electrons of tellurium the band would not be completely filled, and tellurium would possess metallic properties. For the conduction electrons the assumption of a strong spin-orbit splitting does not appear indispensable.

It is interesting to note that the results obtained in Ref. 1 are entirely applicable for lattices of the type $C_{3v}^1 - C_{3v}^6$, for which the group of the wave vector $\mathbf{k} = 0$ is C_{3v}. In this case the contact of the bands along the k_z axis is due to the fact that the group of the wave vector is not changed by displacement along the k_z axis and the point $\mathbf{k} = 0$ is a point of zero slope. The only difference is that in Eq. (33) one must replace all terms of the form Ak_zk_x by Ak_zk_y. A similar treatment can be carried through for the point $\mathbf{k} = 0$ in lattices for which the space groups are related to the point groups C_{4v}, D_{2d}, D_4. In this case $\mathbf{k} = 0$ is a point of zero slope and the zones make contact* along the z axis (for D_{2d} this is a consequence of the symmetry with respect to time reversal[2]). The dispersion law of $E(\mathbf{k})$ is obtained from a secular equation of the form:

$$\begin{vmatrix} (C_1+C_2)k_x^2 + C_4k_y^2 + C_3k_z^2 - E, & (C_1 - C_2 + C_4)k_xk_y \\ (C_1 - C_2 + C_4)k_xk_y, & (C_1+C_2)k_y^2 + C_4k_x^2 + C_3k_z^2 - E \end{vmatrix} = 0. \tag{5}$$

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*For D_4 the zones make contact only in the approximation quadratic in k_z.

For $C_1 = 0$ Eq. (5) is the same as Eq. (33) of Ref. 1 with $A = 0$. Thus the two ellipsoids[3] tangent at the k_z axis are the limiting case of the equipotential surfaces for the whole sequence of space groups $D_{2d}^1 - D_{2d}^{12}$, $C_{3v}^1 - C_{3v}^{12}$, $C_{4v}^1 - C_{4v}^{12}$, $D_4^1 - D_4^{10}$, and this can hold for tellurium type lattices if the relation $p_{1,2}^z = 0$ is fulfilled.

In conclusion I take occasion to thank E. I. Rashba, who called my attention to the illegitimacy of the conclusions drawn in Ref. 1 from the relations (22).

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[2] C. Herring, Phys. Rev. 52, 361 (1937).

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Translated by W. H. Furry
43

POTENTIAL WELLS FOR CHARGED PARTICLES IN A HIGH-FREQUENCY ELECTROMAGNETIC FIELD

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As is well known, in source-free regions of an electrostatic field there can be no absolute maxima or minima of the potential; this fact excludes the possibility of maintaining a charged particle in a state of stable equilibrium (Earnshaw's theorem). This same situation also excludes the possibility of localizing particles, if by localization we mean a state in which a particle with energy smaller than some given magnitude cannot go beyond the limits of a bounded region, no matter what the initial conditions.

The above statement does not apply in the case of a high-frequency electromagnetic field where, as we shall show below, localization of particles can be accomplished.

Consider a particle of charge e and mass m moving in an external electromagnetic field $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})e^{i\omega t}$, $\mathbf{H}(\mathbf{r}, t) = \mathbf{H}(\mathbf{r})e^{i\omega t}$. In the nonrelativistic approximation the equation of motion is

$$\ddot{\mathbf{r}} = \eta \mathbf{E}(\mathbf{r}, t) + (\eta / c)\dot{\mathbf{r}} \times \mathbf{H}(\mathbf{r}, t), \tag{1}$$

where $\eta = e/m$. If the frequency of the external field ω is sufficiently high, the solution of Eq. (1) can be written as a sum of a slowly varying (in terms of the oscillation period of the external field) function $\mathbf{r}_0(t)$ and an oscillating function $\mathbf{r}_1(t)$ (frequency ω). Assuming that $\mathbf{r}_1(t)$ is much smaller than the distance L over which the amplitude of the external field changes markedly,

$$|\mathbf{r}_1(t)| \ll L, \tag{2}$$

and neglecting terms of order $|\mathbf{r}_1/L|$ and $|\dot{\mathbf{r}}_0/L|$, averaging Eq. (1) over the period of the high-frequency field we obtain an equation for $\mathbf{r}_0(t)$:

$$\ddot{\mathbf{r}}_0(t) = -\nabla\Phi, \quad \Phi = (\eta/2\omega)^2 |\mathbf{E}|^2 \tag{3}$$

Thus, the time average of the force acting on the particle is derivable from a potential; the potential is proportional to the square of the modulus of the electric intensity and is independent of the sign of the charge.

There are an unlimited number of possibilities for creating potential wells $\Phi(\mathbf{r})$. The simplest of these are realized in quasi-electrostatic multipole fields

$$\mathbf{E}(\mathbf{r}, t) = \nabla \{r^n P_n^m(\cos\theta)\cos m\varphi\}\, e^{i\omega t},$$

where r, θ, and φ are the spherical ordinates and the P_n^m are the associated Legendre polynomials. For example, the potential Φ in the field of a quasi-static axial quadrupole ($m = 0$, $n = 2$) is of the form $\Phi = \text{const}\, r^2(1 + 3\cos^2\theta)$, i.e., there is an absolute minimum at the origin.*

To determine the motion of the particle inside the potential well we consider the first integral of Eq. (3):

$$\frac{m}{2}\left(|\dot{\mathbf{r}}_0|^2 + \frac{\eta^2}{2\omega^2}|\mathbf{E}|^2\right) = \frac{m}{2}\left(|\dot{\mathbf{r}}_0|^2 + \frac{1}{2}|\dot{\mathbf{r}}_1|^2\right)$$

$$= \text{const} = |e|V_0. \tag{4}$$

The left-hand part of Eq. (4) is equal to the time average of the kinetic energy of the particle, where the kinetic energy of the oscillatory (with frequency ω) motion plays the role of the potential energy.

If the $\mathbf{E} = 0$ at the center of the potential well, particles with energy less than or equal to V_0, are localized within a region at the boundaries of which the following conditions are satisfied

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*It is interesting to note that in an axially-symmetric quadrupole field the original equation (1) in Cartesian coordinates leads to three Mathieu equations; this allows us to analyze the properties of the solution without the limitation imposed by (2).

$$L\omega^2/|\eta| \gg |\mathbf{E}| > 2\omega(V_0/|\eta|)^{1/2}.$$

Finally, we may note in forming potential wells it is possible to make simultaneous use of fields at different frequencies;[1] then, in averaging Eq. (1) over a sufficiently long time interval the potential Φ in Eq. (3) will be of the form

$$\Phi = (\eta^2/4)\sum_n (|\mathbf{E}_n|/\omega_n)^2.$$

Thus it is possible to create three-dimensional potential wells from one-dimensional and two-dimensional wells. This possibility is of interest, in itself, as a means of focussing rectilinear beams of charged particles.[2,3]

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[2] V. S. Tkalich, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 625 (1957), Soviet Phys. JETP **5**, 518 (1957).

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Translated by H. Lashinsky
44

ON THE THEORY OF THE DE HAAS – VAN ALPHEN EFFECT FOR OPEN ISOENERGETIC SURFACES

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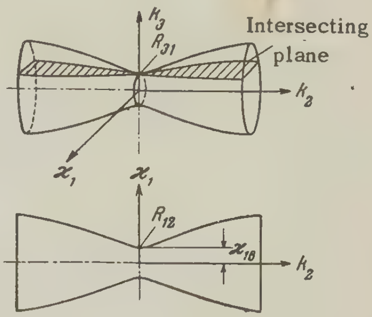
THE theory of electron motion with an arbitrary dispersion law $E(\mathbf{k})$ within a crystal located in a uniform magnetic field was developed in Refs. 1–3. The magnetic susceptibility χ of an aggregate of such electrons oscillates with changes of the magnetic field strength H (the de Haas – van Alphen effect). When the isoenergetic surface in **k**-space that is determined by the equation $E(\mathbf{k}) = \text{const}$ is closed, the period of these oscillations is determined[1,2] by the magnitude of the extremal section S_m of the isoenergetic surface by a plane perpendicular to **H**. When the surface $E(\mathbf{k}) = \text{const}$ is an open surface such as a corrugated cylinder with the magnetic axis perpendicular to the cylinder axis, the oscillations are generally determined not by extremal sections but by the "boundary" sections which are discussed below.

The figure shows part of an open surface and its intersection with the plane $k_3 = \text{const}$ $(H = H_z)$. When k_3 is greater than some value k_b the trajectories (the curves which bound the section) are closed, and when $|k_3| < k_b$ they are open. We use the term "boundary section" for that obtained with the plane $k_3 = k_b$. The energy spectrum of electrons in closed trajectories, i.e., with $|k_3| > k_b$, is almost discrete (slightly broadened discrete levels), and when $|k_3| < k_b$ it is almost continuous (small gaps in a continuous spectrum[3]). Close to the boundary section the width of the gaps is of the same order as the width of the allowed bands (formed from discrete levels[3]).



Because of the (exponential) smallness of the discontinuities (gaps) when $|k_3| < k_b$, the extremal section makes a very small contribution to the oscillating part of the state sum (1). The principal contribution comes from the boundary section.

The number of electron states with energies from zero to E will be[1-3]

$$Z_{\text{osc}}(E, H) = \frac{1}{2\pi^2 a_2}\sum_\alpha \sum_{p=-\infty}^{+\infty}{}' \frac{1}{2\pi i p} \times \int_{-k_{3\,\max}}^{k_{3\,\max}} dk_3 \int_0^{a_1/2\alpha_0^2} \exp\{-2\pi i p n(k_1, k_3, E)\}\, dk_1. \tag{1}$$

Here a_2 is the lattice constant, $\alpha_0 = \sqrt{\hbar c/eH}$, k_1 is a continuous parameter describing the broadening, and α is the spin quantum number. For sections S (in a single cell of **k**-space) close to the boundary section the following dispersion law was obtained in Ref. 3:

$$n = \frac{\alpha_0^2}{2\pi}S - \frac{1}{2} - \frac{1}{\pi}\arcsin\frac{\cos(2\pi\alpha_0^2 k_1/a_2)}{\sqrt{1+e^{-\pi\lambda}}}, \tag{2}$$

$$\lambda = -2a_1^2\varepsilon^{-2}(k_3-k_b)R_{31}\sqrt{R_{12}/\varkappa_{1b}},$$
$$\varepsilon=\sqrt{a_1a_2/\alpha_0}\quad(\varepsilon\ll 1), \tag{3}$$
$$R_{31}^{-1}=-\partial^2k_b/\partial\varkappa_{1b}^2,\quad R_{21}^{-1}=-\partial^2\varkappa_{1b}/\partial k_2^2.$$

Here κ_{1b} is the value of κ_1 on the boundary of cells in **k**-space (for the boundary section κ_{1b} and R_{12} vanish but their ratio remains constant). Substituting (2) in (1), integrating over k_1 and dropping small quantities, we obtain for the integral in (1)

$$\frac{a_2}{2a_0^2}\int\limits_{-k_{3\max}}^{k_{3\max}} e^{-ip\alpha_0^2}\,[e^{(k_b-k_3)/k_0}+1]^{-1}\,dk_3, \tag{4}$$

where

$$k_0=(\varepsilon^2/2\pi a_1^2R_{31})\sqrt{\varkappa_{1b}/R_{12}}. \tag{5}$$

For the calculation of (4) it is necessary to know the dependence of the area of a section S on $x = (k_b - k_3)/k_0$. Assuming that in the vicinity of the "isthmus" shown in the figure the surface (a corrugated cylinder) can be replaced by a one-sheet hyperboloid, we obtain this dependence in the form

$$S=S_b-(1/\pi\alpha_0^2)\,x\ln|ax|;$$
$$a=(\varepsilon^2/4\pi^3)(a_2/a_1)^2\sqrt{R_{12}/\varkappa_{1b}}. \tag{6}$$

The integral (4) with the single parameter $|\ln a|$ was calculated numerically, assuming $|\ln a| \sim 1-10$.

Passing from the state sum to the thermodynamic potential Ω, we obtain

$$\Omega_{\rm osc}=\frac{a_2kT}{\pi a_1\alpha_0^4R_{31}}\sqrt{\frac{\varkappa_{1b}}{R_{12}}}\frac{1}{(\ln a)^2}\left(1+\frac{3\pi}{|\ln a|}\right)$$
$$\times\sum_{p=1}^{\infty}\frac{\cos\left(p\alpha_0^2S_b(\zeta)-\gamma\right)\cos\left(\frac{1}{2}\mu_0H\frac{\partial S_b}{\partial\zeta}p\alpha_0^2\right)}{p^3\sinh\left(\pi p\alpha_0^2\frac{\partial S_b}{\partial\zeta}kT\right)}, \tag{7}$$
$$\gamma=\arctan\frac{3p|\ln a|+9\pi}{p|\ln a|+6\pi}-\frac{\pi}{2},$$

which differs from the usual potential (for closed surfaces) as follows: (1) The period of the oscillations is determined not by S_m but by S_b and the phase γ is weakly dependent on the magnetic field; (2) the factor $|\partial^2S_m/\partial k_3^2|^{-1/2}$ of the amplitude is replaced by

$$\frac{1}{\alpha_0R_{31}}\frac{1}{(\ln a)^2}\left(1+\frac{3\pi}{|\ln a|}\right)\sqrt{\frac{\varkappa_{1b}}{R_{12}}}.$$

This denotes multiplication of the amplitude by $H^{1/2}$, so that the oscillating terms in χ will contain H^{-1} instead of $H^{-3/2}$. In absolute magnitude the amplitude is generally smaller by the factor ϵ^{-1} than for electrons in closed trajectories. But when, for example, we have a field $H \sim 10^4$ oersted and thus $\epsilon \sim 10^{-2}$, the reduction by the factor ϵ^{-1} can be concealed by other factors.

Other oscillatory effects such as oscillations of the resistance (the Shubnikov – de Haas effect), oscillations of the thermoelectric power or of the Hall field, etc. will also be determined in the present case by boundary and not by extremal sections.

The author takes this opportunity to thank I. M. Lifshitz for discussion of this work.

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Translated by I. Emin
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CONCERNING THE THEORY OF RAYLEIGH SCATTERING OF LIGHT IN LIQUIDS

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IN a recently published article, S. M. Rytov[1] develops a theory of scattering of light in liquids under the assumption that this scattering is due entirely to fluctuation deformations $u_{\alpha\beta}$ and to fluctuations of the temperature Θ. Actually, only the fluctuations of $u_{\alpha\beta}$ and Θ are considered in Ref. 1, when it is indicated at the same time that the theory includes all internal processes in the medium and describes the entire spectrum of the scattered light. Next, it is emphasized, in the remarks on p. 518 of Ref. 1 [p. 404 in the translation] that the spectral amplitudes of the fluctuations of any internal parameter can be expressed linearly in terms of $u_{\alpha\beta}$ and Θ, and there is no scattering by isotropic fluctuations which cannot be reduced to density and temperature fluctuations (the possibility of the occurrence of such a scattering was noted by the author[2]).

However, we do not believe this point of view to be correct: scattering of light in the isotropic bodies discussed here does not reduce to fluctuations of $u_{\alpha\beta}$ and Θ. In fact, in the absence of temperature fluctuations and for a fixed position of the centers of gravity of the molecules (i.e., going to the macroscopic treatment as the fluctuations $u_{\alpha\beta}$ go to zero), scattering of light will take place in an ideal gas, consisting of anisotropic molecules, as the result of fluctuations in the orientations of the axes of the molecules. As we go over to dense gases or to liquids, the situation does not change qualitatively, and the scattering will thus take place also when $u_{\alpha\beta} = 0$ and $\Theta = 0$.

In the absence of fluctuations of $u_{\alpha\beta}$ and Θ, it is possible also to have scattering due to the formation of temporary complexes in the liquid (this effect is particularly clearly pronounced in scattering by concentration fluctuations in the case of solutions). This can be readily illustrated quite clearly with models pertaining to the radio-frequency band. Let us consider, for example, small solid hollow non-metallic spheres, with scattering dipoles placed in the centers of these spheres. It is clear that a set of such spheres, no matter what their density, will scatter radial waves as a result of the fluctuations in the orientations of the dipoles, even in the total absence of scattering due to the inhomogeneous spatial distribution of the spheres, or due to other factors. Let us note, furthermore, that the general theory[3] leads to the possible existence of scattering which cannot be described merely by introducing the symmetrical tensor $\epsilon_{\alpha\beta}$; yet, the tensor $u_{\alpha\beta}$, and therefore also the tensor $\epsilon_{\alpha\beta}$ considered in Ref. 1, is symmetrical.*

Under real conditions, in the case of weak absorption, the anti-symmetrical portion of the scattering is apparently quite small.[2] It is possible that this holds in most cases also for scattering by isotropic fluctuations that do not reduce to density and temperature fluctuations. However, we see no grounds for assuming that in low-viscosity liquids the fluctuation shear deformations, $u'_{\alpha\beta} = u_{\alpha\beta} - \frac{1}{3}u_{\gamma\gamma}\delta_{\alpha\beta}$, play a major role in the corresponding portion of the scattering [we have in mind scattering for which $\epsilon_{\gamma\gamma} = 0$; see Ref. 2, where this portion of the tensor $\epsilon_{\alpha\beta}$ is denoted by $\Delta\epsilon'^{(s)}_{ik}$].

It is clear from the above that in Ref. 1, and therefore in Ref. 4, only part of the scattering is actually considered. This circumstance limits even more the possibility of a real utilization of the formulas obtained in Refs. 1 and 4. The point is that experimentally, at a fixed frequency ω_0 of the incident light and at a fixed scattering angle φ, one determines either the two frequency functions $j_x(\Omega)$ and $j_z(\Omega)$, or else their combinations — the total intensity $j(\Omega) = j_x + j_z$, and the degree of depolarization $\Delta(\Omega) = j_x/j_z$. At the same time, Refs. 1 and 4 involve sixteen functions of Ω, namely the complex functions $\overline{K}$, $\overline{\mu}$, κ, C, D, X, Y, and Z. There is no doubt that certain of these functions can be assumed to be practically constant, and others can be approximated in a judicious manner. However, under conditions when all these coefficients $\overline{K}$, $\overline{\mu}$, etc. determine only the previously-unknown portion of the scattering, no way is seen for a reliable separation of this portion and for a comparison of the theory of Refs. 1 and 4 with experiment over the entire region of the Rayleigh line. If we bear in mind, however, the most interesting case, that of the Mandel'shtam-Brillouin doublet in low-viscosity liquids, it is difficult, in all probability, to go here beyond accounting for the dispersion of the velocity of sound, as was done successfully by Fabelinskii[5] (see also Ref. 2).

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*The tensor $\varepsilon_{\alpha\beta}(\omega)$, which serves to describe completely the Rayleigh scattering, can be called the dielectric-constant tensor at $\omega \neq \omega_0$ only by convention (ω is the frequency of the scattered light, $\omega_0 = \omega - \Omega$ is the frequency of the incident radiation). There is therefore no wonder that the tenso $\varepsilon_{\alpha\beta}$ can be non-Hermitian,[3] in spite of the fact that the dielectric-constant tensor is Hermitian (when $\omega = \omega_0$ the tensor $\varepsilon_{\alpha\beta}$ is Hermitian, but even in this case it need not necessarily be symmetrical, as is already clear from the fact that optical activity exists).

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INVERSE BETA PROCESSES AND NONCONSERVATION OF LEPTON CHARGE

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RECENTLY the question was discussed[1] whether there exist other "mixed" neutral particles beside the K^0 mesons,[2] i.e., particles that differ from the corresponding antiparticles, with the transitions between particle and antiparticle states not being strictly forbidden. It was noted that the neutrino might be such a mixed particle, and consequently there exists the possibility of real neutrino ⇄ antineutrino transitions in vacuum, provided that lepton (neutrino) charge[3] is not conserved. In the present note we make a more detailed study of this possibility, in which interest has been renewed owing to recent experiments dealing with inverse beta processes.

Lately there appeared the work of Davis,[4] who used a powerful reactor to study the process of A^{37} formation from Cl^{37} under the influence of neutral leptons. The result of the Davis experiment — a nonzero probability for the process under study — if confirmed, definitely shows that the (strict) law of neutrino charge conservation is not valid. Below we assume that:

(a) the neutrino (ν) and antineutrino ($\tilde{\nu}$) emitted in the processes

$$p \rightarrow n + \beta^+ + \nu, \quad n \rightarrow p + \beta^- + \tilde{\nu}, \qquad (1)$$

are not identical particles;

(b) the strict law of neutrino charge conservation is not valid and consequently processes of the type

$$p \rightarrow n + \beta^+ + \tilde{\nu}, \quad n \rightarrow p + \beta^- + \nu. \qquad (2)$$

are possible, although by definition less probable, than processes (1).

We do not go here into the physical reason for the distinction between neutrino and antineutrino; it might be connected with an approximate conservation law of some quantum number of the type of neutrino charge (analogously to the case of K^0 and $\tilde{K}^0$ mesons, the difference between which is connected with the approximate law of conservation of strangeness).

It follows from the above assumptions that in vacuum a neutrino can be transformed into an antineutrino and vice versa. This means that the neutrino and antineutrino are "mixed" particles, i.e., a symmetric and antisymmetric combination of two truly neutral Majorana particles ν_1 and ν_2 of different combined parity.[5]

The possibility outlined above does not simplify the theory of β decay and, in addition, it probably does not correspond to actuality. Nevertheless, we are setting it forth since it leads to consequences which, in principle, can be tested experimentally. Thus, for example, a stream of neutral leptons consisting mainly of antineutrinos when emitted from a nuclear reactor, will consist at some distance R from the reactor of half neutrinos and half antineutrinos. Under the condition that $R \lesssim 1$ m (the probability for this is discussed below) neutrino experiments, reminiscent of the Pais-Piccioni experiments with K^0 mesons, become possible. Thus, in the experiment of Cowan and Reines,[6] if $R \lesssim 1$ m, when the neutral particles from the reactor are captured by hydrogen, the cross section for formation of neutrons and positrons should be smaller than the cross section expected on simple thermodynamic grounds. This is due to the fact that the stream of neutral leptons, which at creation has a known probability for initiating the reaction, has its composition changed on the way from the reactor to the detector. It would be most interesting to perform the experiment of Ref. 6 at different distances from the reactor. On the other hand, it is difficult to predict the effect of real antineutrino → neutrino transitions on the Davis experiment,[4] for we do not deal here with a strictly inverse β process, and various unknown factors may be important, such as the polarization and energy dependence of the polarization of the neutral leptons from the reactor and from $A^{37} \rightarrow Cl^{37}$ transitions. Therefore, one cannot state a priori, as one could if parity were conserved, that the antineutrino stream, which at creation is essentially unable to initiate the reaction under discussion, is transformed into a stream with a well-defined fraction capable of initiating the reaction. However, one cannot exclude the possibility that the apparent contradiction — the small probability of double β-decay[7] and the relatively large probability of observing A^{37} (Ref. 4) — is partially due to the possibility that in the experiment of Ref. 4, the stream of neutral particles changes its composition on the way from reactor to detector.

The upper limit of R, which can give rise to the mentioned effect in the experiment of Cowan and Reines, is on the order of 1 m, which corresponds to a lifetime $T \lesssim 10^{-8}$ sec for the $\nu \rightleftarrows \tilde{\nu}$ transformation. If one takes into consideration, as pointed out by I. Ia. Pomeranchuk, that the neutrino energy is always at least several orders of magnitude larger than $m_\nu c^2$ (m_ν = neutrino rest mass), and therefore one has in the laboratory system a considerable relativistic dilation of the transformation time, one is faced with the question whether the condition $T \lesssim 10^{-8}$ sec is not completely improbable even under assumptions (a) and (b). The time T is related to the mass difference Δm between the particles ν_1 and ν_2. Δm is proportional to the first power of the matrix element H for the transition $\nu \rightleftarrows \tilde{\nu}$, about which unfortunately nothing definite can be said without some more detailed assumptions regarding the β-process — such as, for example, the Preston scheme,[8] according to which the scalar covariant in the interaction is responsible for the emission of neutrinos, the tensor for the antineutrinos, with comparable although different coupling constants. In that case the transformation $\nu \rightarrow \tilde{\nu}$ is caused by two successive virtual transitions, each of which is characterized by a coupling constant on the order of the constant G of weak interactions ($G \sim 10^{-7} - 10^{-6}$ in units $\hbar = c = \mu = 1$, where μ = π-meson mass); hence H will be proportional to G^2 and $\Delta m \sim 10^{-11} m_e$. The time[9] T turns out to be $\sim 10^{-10} \times$ (neutrino energy)/($m_\nu c^2$) sec, which is considerably longer than 10^{-8} sec.

However, one cannot exclude a direct (first order in G) interaction responsible for the neutrino-antineutrino transformation

$$\nu \rightarrow (\tilde{\nu} + N + \tilde{N}) \rightarrow \tilde{\nu}.$$

In this case Δm is proportional to the first power of the coupling constant,[9] and $T \sim 10^{-16} \times$ (neutrino energy)/($m_\nu c^2$) sec. For a neutrino of 1 Mev energy with $m_\nu c^2 = 100$ ev (experiments[10] show that $m_\nu c^2 \lesssim 500$ ev), one gets $T \sim 10^{-12}$ sec.

In conclusion we wish to emphasize that, independently of the probability of the concrete effects discussed above and of the form of the theory, nonconservation of neutrino charge with distinct neutrino and antineutrino (or, what is the same, the existence of two Majorana neutrinos with different combined parity) inescapably leads to effects of the Gell-Mann — Pais — Piccioni[2] type. The effects due to neutrino-antineutrino transformations may not be observable in the laboratory, owing to the large R but they will take place on an astronomical scale.

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ANGULAR ANISOTROPY IN THE EMISSION OF FRAGMENTS UPON FISSION OF Pu^{239} *BY 14-MEV NEUTRONS*

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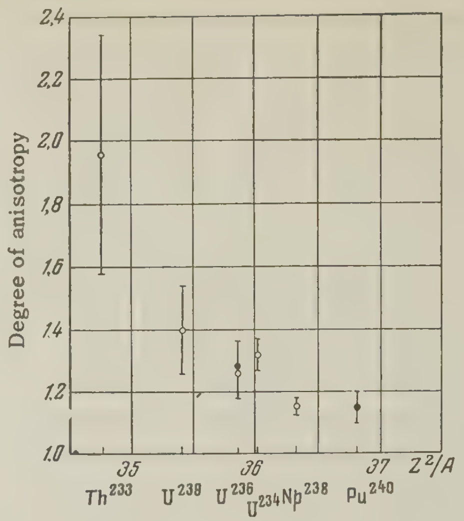
IT was found in previous work[1-5] that the angular distribution of fission fragments, where the fission is induced by fast particles, is anisotropic, and that the extent of the anisotropy (the ratio of the probability of emission of the fragments in directions parallel and perpendicular to the incident particle beam) depends on the nature of the target nucleus.

From considerations developed by A. Bohr,[6] one would expect a large anisotropy in the fast-neutron fission of Pu^{239}, which has a small spin $I = \frac{1}{2}$ and which forms an even-even compound nucleus on capturing a neutron.

An ionization chamber was used to determine the magnitude of the anisotropy in the fission of Pu^{239}. The target of Pu^{239} (with a surface density of about 300 $\mu g/cm^2$) was placed on a collimator whose apertures were drilled at an angle of 45° to its surface. The fission was induced by 14.8-Mev neutrons, while the axis of the neutron beam made an angle of 45° with the surface of the target and the collimator. As in the work of Brolley and Dickinson,[2] a simple rotation of the chamber around its axis (perpendicular to the surface of the collimator) switched the count from the fragments parallel to the neutron beam to the fragments perpendicular to the beam. The maximum angle between the direction of emission of the fragments and the fixed direction (0° and 90°) was 25°. The flux of neutrons on the plutonium target was determined from the number of fission events originating in a layer of U^{235} placed on the back of the plutonium layer.

The degree of anisotropy of Pu^{239} was found to be 1.14. For comparison with the results of Ref. 2, the anisotropy of U^{235} was also determined and was found to be 1.25. The values measured were corrected for the motion of the center of mass, for the finite angular resolution, and for the background of scattered neutrons. Taking the accidental errors in the experiment (mainly random) into account, it was found finally that the extent of the anisotropy is 1.15 ± 0.05 for Pu^{239} and 1.28 ± 0.07 for U^{235}.

The observation of a smaller anisotropy for Pu^{239} than for U^{235} $(I = {}^{7}/_{2})$ is not in agreement with the predictions of A. Bohr[6] and thus shows that the anisotropy of nuclei of a given parity cannot be determined by the spin of the target nucleus alone. This was noted previously by Frank[7] in the analysis of the results of the work of Cohen et al.[5] In connection with this, it is interesting to note that the anisotropy of different nuclei fissioned by 14-Mev neutrons decreases as the parameter Z^2/A increases, as shown in the figure. A similar dependence of the anisotropy on the parameter Z^2/A (for neutron energies of about 7 Mev and for a smaller range of studied nuclei) was also noted by Henkel and Brolley.[4] Unfortunately, the scant and not very accurate experimental data do not permit searching for a more specific dependence on nuclear parameters. Besides, the decrease in the anisotropy with increasing Z^2/A may be related to the large anisotropy in asymmetric fission[5,8] and the reduced relative probability of asymmetric fission with increasing Z^2/A (Ref. 9). In the case of heavier nuclei, the quantum effects discussed in Refs. 6 and 10 may appear in the form of some deviations from the basic tendency toward decreased anisotropy.



○ – Data of Refs. 2 – 4; ● – Data from this work. The parameters Z^2/A of the compound nuclei are shown.

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ON THE THEORY OF SCATTERING OF PARTICLES BY NUCLEI

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THE scattering of particles by nuclei has been investigated using the degenerate Fermi gas as a model for nuclear matter.[1-3] This allows one to evaluate certain effects in scattering linked to the Pauli principle. Assuming that the differential cross section for particles colliding with free nucleons is isotropic, independent of energy and equal to $\sigma_0/4\pi$, it is possible to obtain the total collision cross section for particles against one of the nucleons of the nucleus in the form $\sigma = \sigma_0 F$, where the factor F is a function of two variables: $w = p_F/p$ and $\alpha = m/m_N$. Here, m and p denote the mass and momentum of the particle incident on the nucleus in the laboratory system, while m_N and p_F denote the mass of the nucleon and the momentum on the Fermi surface. The function $F(w, \alpha)$ has already been obtained for $\alpha \leq 1$. The case $\alpha = 1$ which corresponds to scattering of a nucleon against a nucleus has been considered in Refs. 1 and 2, and the case $\alpha < 1$ (meson scattering) in Ref. 3.

An investigation of the region $\alpha > 1$ permits one to construct the function over the whole $w - \alpha$ plane (more exactly, in the first quadrant). The final results are as follows: There are four regions in which F has various analytical forms; the regions are separated from one another by the curves $\alpha w = 1$ and $w = 1$ (see Fig. 1).

$$F = 1 - w^2(1 + 2\alpha)/5 \quad \text{in region I},$$

$$F = [\alpha^2 - 2 + 10w^2 - 20\alpha w^3 + 15\alpha^2 w^4 - 4\alpha^3 w^5] / 10w^3(1-\alpha)^2 \quad \text{in region II},$$

$$F = (1+\alpha)^2 / 10\alpha^2 w^3 \quad \text{in region III},$$

$$F = [1 - 10\alpha^2 w^2 + 10\alpha^2(1+\alpha^2)w^3 - 15\alpha^4 w^4 + 2\alpha^2(3\alpha^2 - 1)w^5] / 10\alpha^2(\alpha - 1)^2 w^3 \quad \text{in region IV}.$$

Figures 2 and 3 show the curves F(w) for values of α which correspond to π, K, N, Ξ, Σ. An analogous curve for Λ falls practically on the Σ curve. In Fig. 4, the abscissa is the energy $\epsilon = E/E_F$ (in units of $E_F \approx 25$ Mev).

The author is grateful to L. B. Okun' for valuable comments.

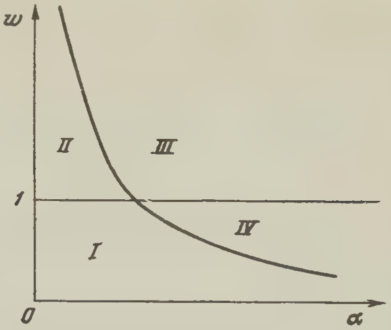

FIG. 1

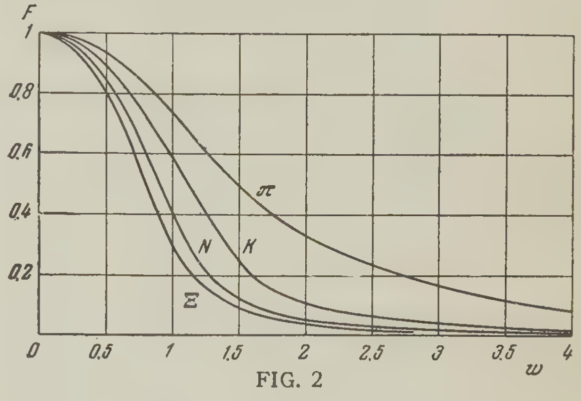

FIG. 2

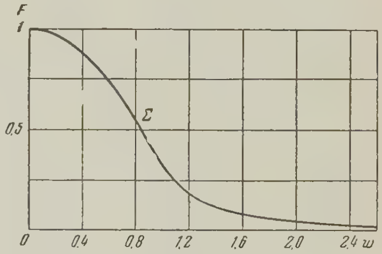

FIG. 3

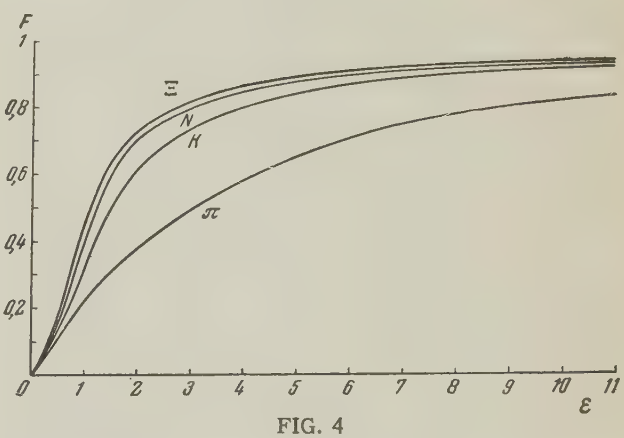

FIG. 4

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ANGULAR DISTRIBUTION OF POSITRONS IN $\pi^+ - \mu^+ - e^+$ DECAY IN PROPANE

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It is known that if parity conservation is violated in the μ–e decay, the angular distribution of the decay electrons can be approximated by a function of the form[1,2]

$$dN \sim \left(1 + \frac{\lambda}{3}\cos\vartheta\right) d\Omega = (1 + a\cos\vartheta)\, d\Omega, \qquad (1)$$

where ϑ is the angle between the direction of motion of the electron (within the range of the solid angle $d\Omega$) and the direction of the μ-meson spin. The experimental angular distributions, obtained by several authors, have a form close to that of the function $(1 + A\cos\vartheta)\,d\Omega$, although the coefficient A is found to be different for various substances.

Actually, in the experiment we measure the angular distribution relative to the direction of motion of the μ mesons, since polarization of μ mesons along their momentum takes place in the π–μ decay. However, a fraction of the μ mesons have the orientation of their spins changed before decaying. The difference in degree of disorientation in different substances is responsible for the variations in value of A and does not permit identification of A with the coefficient a in Eq. (1). A statistically reliable number of measurements of A in various substances must be compiled in order to determine a and to clarify the disorientation mechanism of the μ mesons.

We have studied the angular distribution in π^+–μ^+–e^+ decay in propane. This is valuable, too, considering the usefulness of propane (widely utilized in bubble chambers) for measurements of angular correlations of events of similar nature as in the μ–e decay.

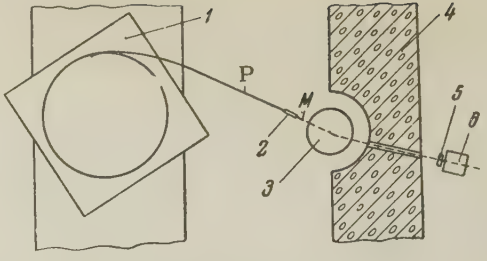

FIG. 1. Setup of the chamber at the accelerator: 1 – vacuum chamber of the accelerator, 2 – target, 3 – deflecting magnet, 4 – shielding, 5 – absorber, 6 – chamber, P – proton beam, M – π^+ mesons.

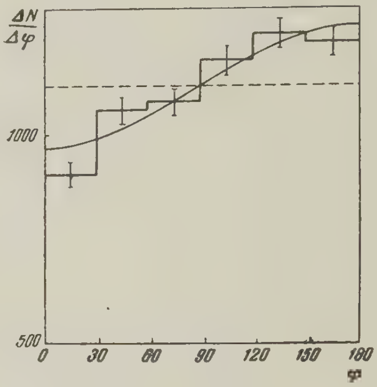

FIG. 2. Electron angular distribution in $\pi^+ - \mu^+ - e^+$ decay. φ is the projection of the spatial angle between the initial directions of the e^+ and μ^+ momenta onto the plane of the camera film.

The experimental setup is shown in Fig. 1. A 7.2 × 6.5 × 16 cm bubble chamber[3] was irradiated by a 175-Mev π^+-meson beam in the synchrocyclotron of the Joint Institute for Nuclear Research. π^+ mesons, formed in a polyethylene target, decayed in the chamber after having been taken out through a collimator and slowed down by a copper absorber. During the measurements, the chamber was inside a magnetic screen, which reduced the external magnetic field to 1 oersted.

In all, we have obtained 8,000 photographs in which we found 6,670 π^+–μ^+–e^+ decays. Not included in this number are decays for which the end of the μ^+ track was closer than 3 mm from the chamber walls. For such borderline cases the probability of observing an electron emitted in the direction of motion of the μ^+ meson could be smaller than if it were emitted backwards.

We have measured the angular distribution of the projection of the spatial angle onto the plane of the camera film. If one assumes an isotropic distribution of the μ^+ mesons in the decay, the function (1) is transformed to read in terms of the plane angle φ as follows:

$$dN \sim [1 + (a\pi^2/16)\cos\varphi]\, d\varphi. \qquad (2)$$

The experimental angular distribution of the decay electrons is shown in Fig. 2. The distribution is satisfactorily approximated by a function of the form (2) (solid line; dotted line represents an isotropic distribution). The ratio of the number of electrons emitted into the angular interval 90 – 180° to the number of electrons emitted into the interval 0 — 90°, relative to the direction of the projection of the initial momentum of the μ^+ mesons, is 1.19. This corresponds to a coefficient $A = -0.22 \pm 0.03$ in the expression $(1 + A\cos\vartheta)$ for the distribution of spatial angles.

As mentioned above, the measured coefficient A is not equal to the coefficient a of Eq. (1). If one denotes by γ the degree of quenching of the μ^+ mesons at the instant of decay, then $A = a \times (1 - \gamma)$. Assuming, as Chadwick et al.[4] do, no disorientation of μ^+ mesons in hydrogen, and using for the determination of $\gamma(C_3H_8)$ the data of Swanson et al.,[5] we find $\gamma(C_3H_8) = 0.33 \pm 0.10$; hence $a = 0.33 \pm 0.06$ and $\lambda = 0.99 \pm 0.18$. The analogous value of λ found by Chadwick et al. from the data in G-5 emulsions equals 0.85 ± 0.18.

We take this opportunity to express our gratitude to Professor V. P. Dzhelepov for making it possible for us to perform this experiment at the synchrocyclotron.

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ENERGY DEPENDENCE OF ANGULAR CORRELATION IN $\mu^- - e^-$ DECAY

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ACCORDING to the two-component neutrino theory[1,2] the angular asymmetry in the μ—e decay depends strongly on the electron energy. If ϵ stands for the ratio of the electron energy to the maximum energy in the μ—e decay, then the angular distribution is described by the function

$$dN = N(a + b\lambda\cos\vartheta)\,d\varepsilon\,d\cos\vartheta, \qquad (1)$$

where ϑ is the angle between the momentum direction of the decay electron and the μ-meson spin; λ is a theoretical parameter;

$$a = 2\varepsilon^2(3 - 2\varepsilon);\quad b = 2\varepsilon^2(2\varepsilon - 1).$$

Several authors, in particular Vaisenberg and Smirnitskii[3,4] have analyzed the angular distribution of positrons of various energies in the π^+–μ^+–e^+ decay and found an increase with energy of the "backward-forward" asymmetry. We have studied the angular correlation in the decay of negative μ mesons, which decayed in an emulsion.

A stack of NIKFI-R photoemulsion 10 cm in diameter and 400μ thick was irradiated by a negative μ-meson beam from the synchrocyclotron of the Joint Institute for Nuclear Research. The μ^- mesons were formed from the decay of 350-Mev π^- mesons and were then separated from other particles by a carbon absorber 90 cm thick. The emulsion chamber was surrounded by a thick layer of iron, which screened it from the stray field of the accelerator.

The geometry of the experiment was such that μ^- mesons of energy close to maximum were registered in the emulsion. The momentum direction of such μ^- mesons changes only slightly in going from the coordinate system in which the π^- meson is at rest to the laboratory coordinate system. For this reason, the μ^--meson beam was considered polarized.

In the scanning of separate emulsion layers tracks of long-range μ^- mesons with decay electrons were noted.

Altogether 630 cases of μ^-—e^- decays were analyzed in which μ^- mesons stopped at a distance of not less than 50μ from any of the emulsion surfaces. In 135 cases of μ^-—e^- decays, an estimate of the electron energy was made when the electron track length was more than 500μ. 83 electrons had a track length of over 1 mm.

The electron energy was measured by the multiple-scattering method; the error in the energy determination ranged from 30 to 18%, depending on the electron track length.

In order to compare the experiment with formula (1) and the consequences resulting from it, it was necessary to measure the angle ϑ, which in our case was taken to be the same as the angle between the direction of the electron motion and the axis of the μ^--meson beam.

In 135 cases of μ^-—e^- decays, 64 electrons were emitted forwards (i.e., $0 \le \vartheta \le 90°$) and 71 electrons were emitted backwards (i.e., $90° \le \vartheta \le 180°$). This circumstance may serve as an indication that the spin direction of the μ^- meson,

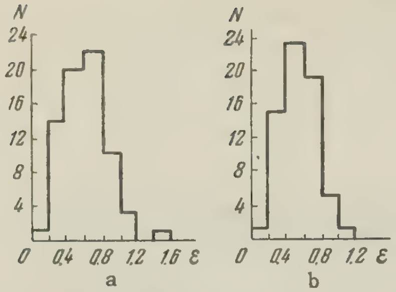

FIG. 1 Electron energy spectra: a – electrons emitted backwards, b – electrons emitted forwards.

at least for some particles, keeps its initial direction in the emulsion up to the instant of decay.

In Fig. 1 the energy spectrum is shown separately for the electrons emitted forwards and backwards. Here the number of electrons N is plotted against the ratio ϵ of the electron energy to the maximum energy $E_{max} = 55$ Mev.

Angles	Direction	Electron Number	Theoretical ratio ($\lambda = -1$)
0–20°	forward	3	1 : 2.6
160–180°	backward	9	
0–60°	forward	11	1 : 2
120–180°	backward	19	
60–90°	forward	8	1 : 1.2
90–120°	backward	13	

For energies $\epsilon > 0.6$ the "forward-backward" electron ratio was 25 : 36. In the low energy region ($\epsilon < 0.6$), 39 particles were emitted forwards and 35 backwards. The observed change in sign of the asymmetry for fast and slow electrons is in qualitative agreement with formula (1).

In addition, we have studied the asymmetry in various angular intervals. The table gives the ratio of "forward-backward" electrons for various angular intervals in the energy region > 35 Mev.

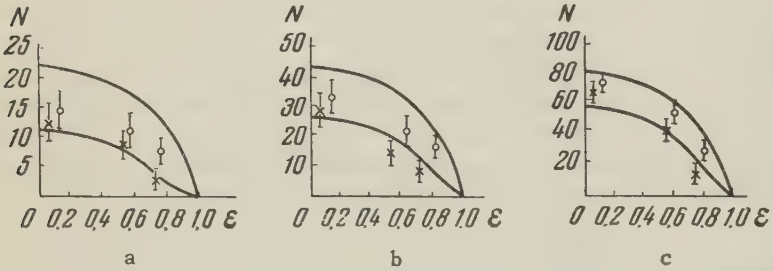

FIG. 2. Electron energy spectra in various angular intervals: a) 0 – 20° and 160° – 180°, b) 0 – 45° and 135° – 180°, c) 0 – 90° and 90° – 180°; o – backward emitted electrons, × – forward emitted electrons.

In Fig. 2 the energy spectra in various angular intervals are compared with theoretical curves. The upper curve is for electrons emitted backwards, and the lower curve for electrons emitted forwards. The slight disagreement between the experimental points and the expected distribution is, apparently, due to a partial depolarization of the μ^- mesons before decaying.

From the above given analysis of the angular correlation in the μ^-–e^- decay one can see a qualitative agreement with the two-component neutrino theory. Unfortunately, the accumulated statistics are not sufficient for quantitative conclusions.

The authors are grateful to A. I. Alikhanian, corresponding member of the U.S.S.R. Academy of Science, for valuable advice, to A. O. Vaisenberg and V. A. Smirnitskii for discussion of results, and also to T. V. Streltsov and G. I. Polosin for help in measurements and scanning of the emulsions.

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51

SCATTERING OF NEUTRINOS BY ELECTRONS

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IMPROVEMENTS in experimental techniques have made it possible to determine cross sections on the order of $10^{-44}-10^{-45}$ cm^2 in reactions dealing with the absorption of neutrinos by protons or Cl^{37} (see Refs. 1 and 2), and have led to the hope that these experiments will be repeated in the near future for the purpose of measuring the scattering of neutrinos by electrons. Up to now, experiments carried out to determine the magnetic moment of the neutrino have led to negative results. According to Ref. 3, σ_μ, the cross section for ν—e scattering, is less than 7.5×10^{-40} cm^2/electron, which leads to a value of μ, the magnetic moment of the neutrino, of less than $10^{-7}\mu_B$ (μ_B is the Bohr magneton). On the other hand, theoretical estimates of Thirring and Houtermans[4] yield $\mu \sim 10^{-10}\mu_B$.

The purpose of the present note is to show the possibility of ν—e scattering through a direct 4-fermion interaction $\bar{e}e\nu\bar{\nu}$. The cross section for such scattering may be of the same order of magnitude as σ_μ and can even exceed it. If the incident neutrino has energy E, and the recoil electron has energy W (in units of mc^2), the scattering cross section has the form

$$d\sigma_E(W)=\frac{m^2}{8\pi\hbar^4}\frac{dW}{E^2}\Big\{|g_S|^2(W^2+WE)+g_P^2W^2$$
$$+2|g_V|^2[W^2-W(2E+1)+2E^2]$$
$$+2|g_A|^2[W^2-W(2E-1)+2E^2]$$
$$+2|g_T|^2[W^2-W(4E+1)+4E^2]$$
$$+2\operatorname{Re}(2g_Vg_A^*+g_Sg_T^*+g_Pg_T^*)(W^2-WE)\Big\}. \qquad (1)$$

The total effective cross section

$$\sigma_f=\int_0^\infty\rho(E)\,dE\int_0^{2E^2/(2E+1)}d\sigma_E(W) \qquad (2)$$

is obtained by averaging over the energy distribution of the incident neutrinos $\rho(E)$. Assuming as in Ref. 5 that $\rho(E)\sim\exp[-E^2/2(\Delta E)^2]$ we find for $\Delta E = 3.8$

$$\sigma_f=0.05\,(m^2/\hbar^4)\,g^2,$$
$$g^2=|g_S|^2+0.5\,|g_P|^2+5\,|g_V|^2+6\,|g_A|^2+$$
$$+9\,|g_T|^2-2.5\operatorname{Re}(2g_Vg_A^*+g_Sg_T^*+g_Pg_T^*). \qquad (3)$$

If one assumes the usual value of the universal Fermi interaction constant $g = 3 \times 10^{-49}$ erg-cm^3, then

$$\sigma_f=3.5\cdot 10^{-45}\ \text{cm}^2. \qquad (4)$$

This value is 5 times larger than the cross section $\sigma_\mu \sim 7.5\times 10^{-46}$ cm^2, corresponding to $\mu \sim 10^{-10}\times\mu_B$.

An interaction of this type may generally also take place in the scattering of a neutrino by a nucleon. Formula (1) may be used as above to obtain a cross section of the same order of magnitude, however, it is practically impossible to observe ν—n and ν—p scattering because of the small recoil energy of the nucleons.

Nonconservation of parity leads to modifying (1) and (3) as follows:

$$g_ig_j^*\rightarrow g_ig_j^*+g_i'g_j'^*-\eta\,(g_ig_j'^*+g_i'g_j^*),$$

where η is the longitudinal polarization of the neutrino beam.

In conclusion, the author wishes to thank I. M. Shmushkevich, V. N. Gribov, and I. T. Diatlov for discussion of this analysis.

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Translated by M. A. Melkanoff
52

EFFECT OF HYDROSTATIC COMPRESSION ON THE ELECTRICAL CONDUCTIVITY OF METALS AT LOW TEMPERATURES

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IN our previous work on the effect of isotropic compression on the superconductivity of tin and indium,[1] it was noted parenthetically that there was a difference in conductivity of these metals when under pressure and in the normal state. In succeeding work this has been confirmed.[2]

Very recently we have also observed that hydrostatic compression has a considerable effect on the magnetic[3] and galvanomagnetic properties of metals[4,5] at low temperatures. From these later studies we conclude that the density of the conduction electrons is noticeably affected by pressure.

All of the above investigations were carried out by the use of a freezing technique to produce the pressure.[6] It is well known that this method can produce pressures up to a maximum of 1730 kg/cm^2, and the large effect of such a relatively small pressure upon almost all of those properties of metals depending on the conduction electrons was very interesting and unexpected. It is therefore essential to study in detail the effect of pressure on the electrical conductivity over a wide range of temperature.

In this letter we shall report some results of studies on the effect of pressure on the conductivities of metals in the low temperature region. Measurements have been made on zinc, tin, gold, and bismuth. All samples (except for gold) were prepared as single crystals. Zinc was studied in two orientations (in sample Zn I the principal axis was perpendicular to the sample axis, and in Zn II the two axes were parallel). The principal axis of symmetry of the tin sample was parallel to the long axis of the sample, and the axes of the bismuth were oriented at about 45° from the length of the sample. The metals were very pure; the ratio of the resistance at 4.2° K to the resistance at 293° K, which characterizes the purity, was 5.3×10^{-4} for the Zn I sample, for example, 6×10^{-5} for the tin, and 3.25×10^{-3} for the gold. Measurements were made at the temperatures of liquid helium, hydrogen, and nitrogen, and also at higher temperatures in the case of bismuth.

The bismuth was studied only in order to compare our results with those of Alekseevskii and his co-workers.[4] In our measurements, as well as theirs, pressure caused an increased resistivity over the whole temperature range for this metal.

The other metals which we investigated behaved in a different way. At sufficiently low temperatures, all of them showed increased resistance under pressure. At higher temperatures, this resistance increment decreased, and at some temperature (peculiar to each metal) it became equal to zero. Upon further raising the temperature, the effect changed sign — the resistance under pressure became less than the resistance without pressure, thus agreeing with the results of Bridgman's work[7] at temperatures above 90° K.

In zinc (sample Zn I) at 4.2° K the increment of resistance due to the pressure amounted to $\Delta R/R \sim 30\%$. The effect decreased with rise in temperature and had almost disappeared at 20.4° K. At 77° K the effect was already reversed in sign, and its magnitude was approaching Bridgman's value for monocrystalline zinc with this orientation at a temperature of 90° K ($\Delta R/R \sim 1.5\%$).

In tin at 4.2° K the resistivity under pressure was also larger than normal — $\Delta R/R$ was about 30%. But at 14° K the sign of the effect had reversed; i.e., the sign change for the effect in tin occurs somewhere between 4.2° and 10° K.

For gold at 4.2° K the effect amounts to about 11%; its reversal takes place in the liquid hydrogen temperature region. As already indicated, the gold sample was quite pure ($R_{4.2}/R_{293} = 3.25 \times 10^{-3}$); its resistivity did not go through a minimum at any temperature, as it does when gold contains impurities, and neither did the resistivity of the sample under pressure.

Thus for all the metals studied in this work, an increase in resistivity was observed when hydrostatic pressure was applied. The effect is reversible.

For the time being, it does not seem possible to explain this effect; but its mechanism is probably quite different from that responsible for the effect of pressure on electrical resistivity at higher temperatures. It is possible that the low-temperature increase of resistivity with pressure has the same origin as the effect of pressure on the magnetic and galvanomagnetic properties of the metals.

It is interesting that the effect should disappear at higher temperatures. In this case the pressure-induced change in conductivity follows an exponential law of dependence on the temperature very closely. For example, in the range 14 to 20° K, the conductivity of sample Zn I was given with good approximation by $\Delta\sigma \propto e^{-\Delta E/kT}$ with $\Delta E = 2 \times 10^{-2}$ ev/mole. Bismuth behaves in a similar manner.

This gives some grounds for supposing that pressure creates a condition similar to that in a semiconductor, for some of the electrons; but with a very small energy gap, which would overlap even at low temperatures.

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53

EFFECT OF μ^--MESON POLARIZATION ON THE CORRELATION OF GAMMA RAYS EMITTED BY THE MESONIC ATOM

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THE function describing the angular correlation of γ rays from mesonic atom transitions was derived in Ref. 1. Recently it has been established experimentally[2] that polarized μ^- mesons are formed in the π^--meson decay. It is therefore of interest to discuss the same question for a given degree of polarization of the μ^- meson.

In the case of heavy mesonic atoms, the correlation function for the cascade $\ell_A (L_1) \ell_B (L_2) \ell_C$ has the form:

$$W = \sum_{\nu_1\nu_2} F_{2\nu_1}(L_1 l_A l_B)\, F_{2\nu_2}(L_2 l_C l_B)$$

$$\times \sum_{F_B F'_B j_\beta l k \rho} \frac{(2F_B+1)(2F'_B+1)}{2I+1) \quad 1+(\nu_{FF'}\tau)^2]} W^2(j_\beta I 2\nu_2 F_B;\ F'_B j_\beta)$$

$$\times (2j_\beta+1)^2 W(l_B s 2\nu_2 j_\beta;\ j_\beta l_B)\, W(l_B s 2\nu_2 j_\beta;\ j_\beta l)$$

$$\times (2l+1)\, W(s2kj_\beta l;\ s l_B)\, W(2\nu_1 2\nu_2 l_B l;\ 2k l_B)\, C^{2k0}_{2\nu_1\rho;\ 2\nu_2-\rho}$$

$$\times <|(ss)\,2k0> 4\pi Y^{\rho}_{2\nu_1}\left(\theta_1,\ \varphi_1-\frac{\pi}{2}\right) Y^{-\rho}_{2\nu_2}\left(\theta_2,\ \varphi_2-\frac{\pi}{2}\right); \tag{1}$$

$$F_{2\nu_1}(L_1 l_A l_B) = (-1)^{l_A-l_B-1}(2l_B+1)^{1/2}(2L_1+1)$$

$$\times C^{2\nu_1 0}_{L_1 1;\ L_1-1} W(l_B l_B L_1 L_1;\ 2\nu_1 l_A); \tag{1a}$$

$$<|(ss)\,2k0> = W^{-1}_{2k}(s)\, f_{2k}(s) = \sum_{\mu} a_\mu C^{2k0}_{s\mu;\ s-\mu}. \tag{1b}$$

Here L_1, L_2 are the angular momenta of the quanta; ℓ_A, ℓ_B, ℓ_C are the orbital angular momenta of the meson in the initial, intermediate, and final states; j_α, j_β, j_γ are the corresponding total angular momenta: $\mathbf{j} = \boldsymbol{\ell} + \mathbf{s}$ where s is the meson spin; F_B is the sum of the meson (j_β) and nucleus (I) angular momenta; $<|(ss)\,2k0>$ is the statistical tensor of Fano, proportional to the meson orientation degree of 2k-th order $f_{2k}(s)$;[3] a_μ is the probability amplitude that the meson in the initial state has a spin projection equal to μ; $\nu_{FF'}$ is the hyperfine structure; τ is the lifetime of the intermediate state; C and W are the Clebsh-Gordan and Racah coefficients.

As can be seen from expression (1) the correlation function depends on the angles formed by the axis of rotational symmetry of the μ meson and the directions of emission of the first (θ_1) and the second (θ_2) quanta, as well as on the angle θ between the directions of the two quanta:

$$\cos\theta = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2 \cos(\varphi_2 - \varphi_1). \tag{1c}$$

If $a_\mu = \text{const} = 1/(2s+1)$ then (1) reduces to formula (2) obtained in Ref. 1.

Since the summation in (1) is only over even orders of the orientation degree $f_{2k}(s)$ we shall also obtain the same result if we set $s = \frac{1}{2}$.[2] Thus the correlation does not depend on the degree of the μ^--meson polarization $f_1(s)$. If one allows for the possibility of the μ^--meson spin being $3/2$ then in this case the correlation function would depend on the degree of alignment $f_2(s)$.

I am grateful to K. A. Ter-Martirosian for his interest in this work.

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54

ON THE THEORY OF MAGNETIC SUSCEPTIBILITY OF METALS

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RECENTLY, the magnetic susceptibility of an electron gas was calculated by several authors,[1-3] taking into account long-range Coulomb correlation. However, only the susceptibility due to the Fermi branch of the excitation spectrum was taken into account there. We would like to direct attention to the fact that the (Bose) plasma-oscillation quanta also make a definite contribution to the susceptibility. Actually, although these excitations are neutral and give no contribution to the current, their energy depends on the magnetic field intensity H and consequently the plasma quanta are "carriers of magnetism." At ordinary temperatures, real plasma quanta in a metal are practically not excited; their zero-point energy, however, also depends on H. This leads, as we shall show, to plasma diamagnetism comparable to the Landau diamagnetism.

As is known (see, for example, Ref. 4), a separation of plasma oscillations into longitudinal and transverse is still possible in a weak magnetic field. For our problem, only the former are of interest; the frequency of a longitudinal plasma quantum (in the frame of an isotropic model) is

$$\omega^2 = \omega_L^2 + \omega_H \sin^2 \alpha + O(k^2), \tag{1}$$

where k is the wave vector of a plasmon, α is the angle between k and H, $\omega_L^2 = 4\pi n e^2/m$, $\omega_H = eH/mc$, and n and m are the concentration and effective mass of the electrons ($\omega_H \ll \omega_L$). We shall disregard terms of order k^2 in Eq. (1) (apparently, they are small in comparison with ω_L^2 for all k up to the limiting wave number k_0).

The magnetic susceptibility per unit volume, due to the dependence of the zero-point energy of the plasma on the magnetic field, is:

$$\chi = -\frac{1}{2}\frac{\hbar}{(2\pi)^3}\frac{\partial^2}{\partial H^2}\int\limits_{k\leqslant k_0} d\mathbf{k}\,\omega(\mathbf{k}). \tag{2}$$

By virtue of Eq. (1), this yields

$$\chi = -(1/18\pi^2)(e^2/mc^2)(\hbar/m)\,k_0^3/\omega_L. \tag{3}$$

The quantity k_0 in our approximation (small H) can be regarded as independent of the magnetic field. Setting $\hbar k_0 = \beta p_F$, where p_F is the Fermi boundary momentum and β is a dimensionless parameter (which may depend on n) we obtain

$$\begin{aligned}\chi &= -(\beta^3/12\sqrt{\pi})(\hbar/mc)(ne^2/mc^2)^{1/2}\\ &= -0.96\cdot 10^{-18}(m_0/m)^{3/2}\beta^3\sqrt{n}\end{aligned} \tag{4}$$

(m_0 is the mass of a free electron). Inasmuch as $\beta < 1$, but evidently $\beta > \frac{1}{2}$,[5] $|\chi| \sim 10^{-6} - 10^{-7}$. This quantity can be fully comparable with the result of Pines, obtained in disregarding the zero-point energy of the plasma. Hence it is clear that this neglect, generally speaking, is by no means justified and the quantiative comparison of Pines' theory with experiment must be reviewed.

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55

THE PROPERTIES OF THE GREEN FUNCTION FOR PARTICLES IN STATISTICS

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IN an attempt to apply the methods which have recently been developed in quantum electrodynamics to statistics, one is faced with the problem of the properties that will be exhibited in that case by the fundamental quantities entering into these methods. We shall show here that some very general relations can be derived for the particle Green function G, which is defined in the well known manner as

$$G = -i\langle T\psi_1\psi_2^+\rangle, \tag{1}$$

where the indices 1 and 2 indicate that the ψ operator must be taken at t_1 or t_2. T is the symbol for the chronologically ordered product, while the averaging is taken over the actual state of the given macroscopic system. It is well known that chronological ordering means that

$$G = -i\langle\psi_1\psi_2^+\rangle \text{ for } t_1 > t_2, \quad G = \mp i\langle\psi_2^+\psi_1\rangle \text{ for } t_1 < t_2, \tag{2}$$

where the upper and lower signs refer to the Bose and Fermi statistics respectively.

The space-time dependence of the matrix elements of the operator ψ is given by equations of the form

$$\psi_{nm}(t, \mathbf{r}) = \psi_{nm}^{(0)} \exp\{i(\omega_{nm}t - \mathbf{k}_{nm}\mathbf{r})\},$$

$$\omega_{nm} = (E_n - E_m)/\hbar, \quad \mathbf{k}_{nm} = (\mathbf{P}_n - \mathbf{P}_m)/\hbar,$$

where the indices n, m refer to the states of a closed system with total energy E and total momentum $\mathbf{P}$. From the definition of the adjoint operator, we have $(\psi^+)_{mn} = (\psi_{nm})^*$.

Using these matrix elements we can write (2) in the form

$$\text{for } t_1 > t_2\text{:} \quad G = -i\sum_m |\psi_{nm}^{(0)}|^2 \exp\{i\omega_{nm}t - i\mathbf{k}_{nm}\mathbf{r}\},$$

$$\text{for } t_1 < t_2\text{:} \quad G = \mp i\sum_m |\psi_{mn}^{(0)}|^2 \exp\{i\omega_{mn}t - i\mathbf{k}_{mn}\mathbf{r}\}, \tag{3}$$

where we use the notation $t = t_1 - t_2$, $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and where the index n refers to the given state over which the averaging of Eqs. (1) or (2) is performed.

We shall now proceed to average expressions (3) over a Gibbsian ensemble. According to the basic principles of statistics, this operation means that we express the quantity G as a function of the temperature T and the chemical potential μ instead of as a function of the energy E and the number of particles in the system N. We have for $t > 0$,

$$G = -i\sum_{n,m} \exp\{(\Omega + \mu N_n - E_n)/T\}$$

$$\times |\psi_{nm}^{(0)}|^2 \exp\{i\omega_{nm}t - i\mathbf{k}_{nm}\mathbf{r}\}$$

where the temperature is measured in energy units. Since we sum now over the two indices n and m we can interchange the indices in such a sum. We use this possibility in the expression for G for $t_1 < t_2$ and write for $t < 0$,

$$G = \mp i\sum_{n,m} \exp\{(\Omega + \mu N_m - E_m)/T\}$$

$$\times |\psi_{nm}^{(0)}|^2 \exp\{i\omega_{nm}t - i\mathbf{k}_{nm}\mathbf{r}\}$$

$$= \mp i\sum_{m,n} \exp\{(\Omega + \mu N_n - E_n)/T\}$$

$$\times \exp\{(\hbar\omega_{nm} + \mu)/T\}\, |\psi_{nm}^{(0)}|^2 \exp\{(i\omega_{nm}t - i\mathbf{k}_{nm}\mathbf{r}\}.$$

In the last transformation we have used the fact that the matrix elements $\psi_{nm}^{(0)}$ are different from zero only if $N_m = N_n + 1$.

We now go over from the space-time representation of the Green function to its Fourier coefficients,

$$G(\omega, \mathbf{k}) = \iint G(t, \mathbf{r})\, e^{i(\omega t - \mathbf{k}\mathbf{r})}\, dt\, d^3\mathbf{r}.$$

Integration over space gives a delta function of $\mathbf{k} - \mathbf{k}_{mn}$. The integration over dt must be performed separately over the interval from $-\infty$ to 0 and from 0 to $+\infty$, using the well known formula

$$\int_0^\infty e^{i\alpha x}\, dx = \pi\delta(\alpha) + \frac{i}{\alpha}.$$

We get as a result

$$G(\omega, \mathbf{k}) = -(2\pi)^3 \sum_{n,m} \exp\{(\Omega + \mu N_n - E_n)/T\}$$

$$\times |\psi_{nm}^{(0)}|^2\, \delta(\mathbf{k} - \mathbf{k}_{mn})\, \{i\pi\delta(\omega - \omega_{mn})\, [1 \pm e^{(\mu - \hbar\omega_{mn})/T}]$$

$$+ \frac{1}{\omega_{mn} - \omega}[1 \mp e^{(\mu - \hbar\omega_{mn})/T}]\}. \tag{4}$$

Comparing the two terms within the braces we see that there exists a certain relation between the real (G') and the imaginary (G'') part of the Green function. In the case of Bose statistics this relation is

$$G'(\omega, \mathbf{k}) = \frac{1}{\pi}\int_{-\infty}^{\infty} \tanh\frac{\hbar x - \mu}{2T} \cdot \frac{G''(x, \mathbf{k})}{x - \omega}\, dx, \tag{5}$$

where we take the principal value of the integral. We have always

$$G''(\omega, \mathbf{k}) < 0. \tag{6}$$

as follows from Eq. (4).

In the case of Fermi statistics we have

$$G'(\omega, \mathbf{k}) = \frac{1}{\pi}\int_{-\infty}^{\infty} \coth\frac{\hbar x - \mu}{2T} \cdot \frac{G''(x, \mathbf{k})}{x - \omega}\, dx, \tag{7}$$

where the sign of $G''(\omega, \mathbf{k})$ is the opposite of the sign of the difference $\hbar\omega - \mu$ and both these quantities go through zero at the same time,

$$G''(\omega, \mathrm{k})/(\hbar\omega - \mu) < 0. \tag{8}$$

At the absolute zero of temperature both (5) and (7) go over into

$$G'(\omega, \mathrm{k}) = \pm\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{G''(x, \mathrm{k})}{x - \omega}\,dx, \tag{9}$$

where the plus sign refers to $\hbar x > \mu$ and the minus sign to $\hbar x < \mu$.

It is interesting to note that these formulae show that the function G is not an analytical function of the variable ω. We can construct two analytical functions (which have no singularities in the upper half-plane) as follows

$$G' + i\tanh\frac{\hbar\omega - \mu}{2T}\cdot G'' \text{and}\, G' + i\coth\frac{\hbar\omega - \mu}{2T}\cdot G''.$$

Translated by D. ter Haar
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